Knot Floer Homology, Folding Automata, and the Connect-Sum of Trefoils

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Two years ago, at BC...

Braeden Reinoso closed his defense by posing:

Open problem (Reinoso, 2024)

Prove that $\widehat{HFK}$ detects $T_{2,3}\#T_{2,3}$ (the granny knot), or find a counterexample to show it does not.

Today: how this story ended.

Main result

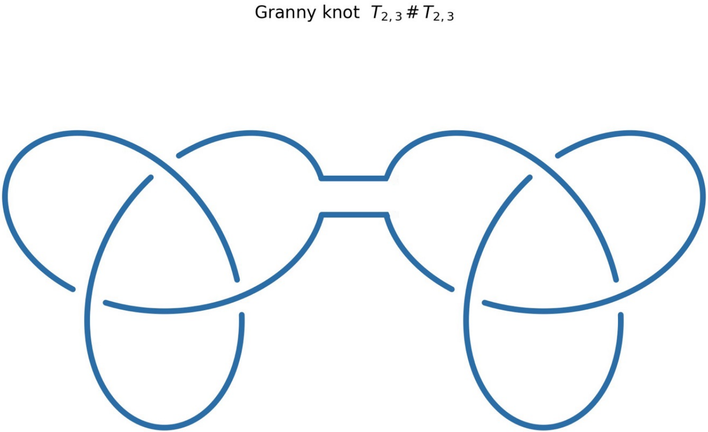
Main Theorem

Knot Floer homology ($\widehat{HFK}$) detects $T_{2,3} \# T_{2,3}$.

- First detection result in $\widehat{HFK}$ of a composite knot.

Building on Ozsváth–Szabó, Ni, Baldwin–Hu–Sivek,
Farber–Reinoso–Wang, Reinoso

The granny knot, $T_{2,3} \# T_{2,3}$



Knot invariants and detection

- Knot invariant — a rule assigning to each knot K a mathematical, often algebraic object (polynomial, group, vector space, etc.).

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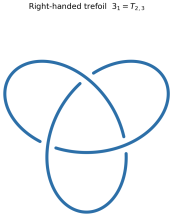
- Knot Floer homology $\widehat{HFK}$ is a knot invariant. The output is a bigraded vector space.
- Detection is rare! "Classical" invariants not known to detect even the unknot.
- Knot Floer homology is known to detect: unknot, trefoils, figure-eight, $T_{2,5}$, $T_{-2,5}$ $\ldots$

Prime vs composite knots

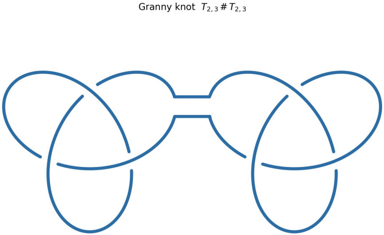
A knot is composite if it splits as a *connect-sum* $K = K_1 \# K_2$ of two non-trivial knots. Otherwise it is prime.

Prime



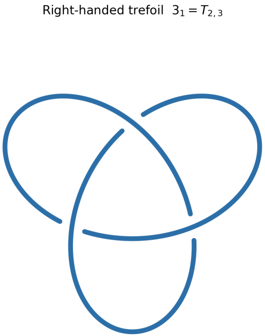
right-handed trefoil $T_{2,3}$

Composite



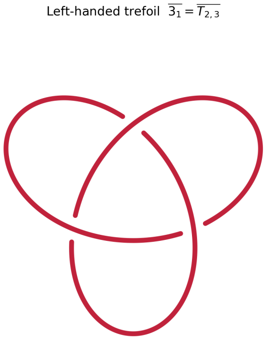
connect-sum $T_{2,3} \# T_{2,3}$, the granny knot

Two trefoils



Right-handed trefoil $3_1 = T_{2,3}$

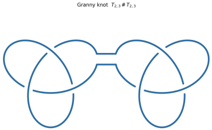
right-handed trefoil, $T_{2,3}$



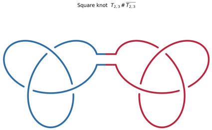
Left-handed trefoil $\overline{3_1} = \overline{T_{2,3}}$

left-handed trefoil $\overline{T_{2,3}}$

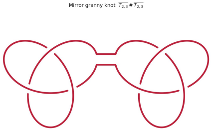
Three connect-sums of two trefoils



Granny $T_{2,3}\#T_{2,3}$



Square $T_{2,3}\#\overline{T}_{2,3}$



Mirror granny $\overline{T}_{2,3}\#\overline{T}_{2,3}$

All three share $\Delta_K(t) = (t^2 - t + 1)^2$.

Knot Floer homology (Ozsvath–Szabo, Rasmussen, around 2003)

- $\widehat{HFK}(K;\mathbb{F}_2)$ is a bigraded $\mathbb{F}_2$-vector space, with Alexander grading A and Maslov grading M.

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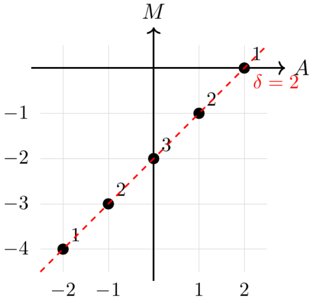
The granny knot:

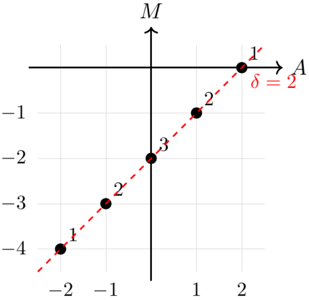


Main result, formally

Main Theorem

If $\widehat{HFK}(K;\mathbb{F}_2) \cong \widehat{HFK}(T_{2,3}\#T_{2,3};\mathbb{F}_2)$ as bigraded vector spaces, then $K \cong T_{2,3}\#T_{2,3}$.

- Feed an arbitrary knot K into the blackbox $\widehat{HFK}$.
- If the output is exactly this bigraded grid *(right)* $\Rightarrow$ K was the granny knot.



$\widehat{HFK}(T_{2,3}\#T_{2,3};\mathbb{F}_2)$

$\widehat{HFK}$ detects fiberedness

Theorem (Ozsváth–Szabó; Ni; Ghiggini; Juhász)

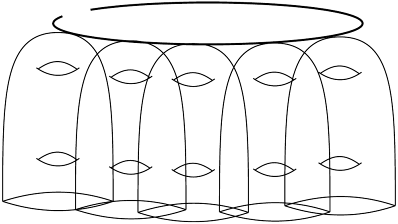
$\widehat{HFK}$ detects both Seifert genus ($g(K) = \max\{A : \widehat{HFK}(K, A) \neq 0\}$) and fiberedness ($K$ fibered $\iff$ $\dim \widehat{HFK}(K, g(K)) = 1$).

Hence: K with HFK matching $T_{2,3} \# T_{2,3}$ is fibered of genus 2.

What does fibered mean?

A knot is *fibered* if its exterior is foliated by an S^1-family of Seifert surfaces. Equivalently: K

is the binding of an *open book*, with pages a genus-g Seifert surface fanned around the binding.



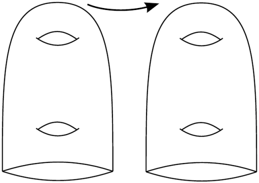
Fibered knot → surface map

As the Seifert surface sweeps around, it foliates the knot exterior — and after one full turn returns to itself via a homeomorphism

$$h : \Sigma \to \Sigma, \quad h|_{\partial\Sigma} = \mathrm{id}.$$

h is the monodromy. The

pair (Σ, h) determines K up to isotopy.



⇒ *reduce the problem:* knowing K is fibered of genus-2, study the monodromy h, argue it can only be the monodromy of $T_{2,3} \# T_{2,3}$.

Mapping class group of a surface

- h is well-defined up to isotopy rel boundary and conjugation. The natural setting:

$$\mathrm{Mod}(\Sigma) \;=\; \mathrm{Homeo}^{+}(\Sigma)\,/\,\text{isotopy}.$$

- Our monodromy: $[h] \in \mathrm{Mod}(\Sigma_2^1)$.
- Restated goal: instead of asking “which knot has the granny’s HFK?”, we now ask: “which mapping class on the genus-two surface could be the monodromy of K?” Same question. Easier object.

Nielsen–Thurston classification

Theorem (Thurston)

Every knot K in S^3 is exactly one of the following:

- Torus knot
- Satellite knot
- Hyperbolic knot

For a fibered knot K, in each case the monodromy h is freely isotopic to a representative ϕ_h:

Theorem (Nielsen–Thurston)

- K is a torus knot $\Rightarrow \phi_h$ is *periodic*;
- K is a satellite $\Rightarrow \phi_h$ is fixes a non-empty set of multicurves;
- K is hyperbolic $\Rightarrow \phi_h$ is *pseudo-Anosov*.

Roadmap

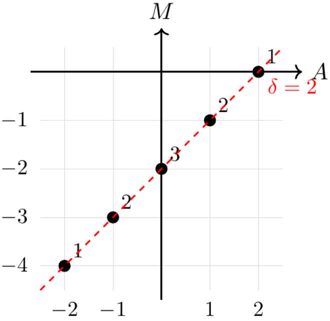
Three cases for our fibered genus-2 knot K:

Knot class	**Monodromy**	**Outcome**
Torus	periodic	ruled out by HFK rank
Hyperbolic	pseudo-Anosov	most of the talk, ruled out by train tracks
Satellite	reducible	forces connect-sum of trefoils

Torus knot case

K is not a torus knot

- Torus knots are known to satisfy $\dim \widehat{HFK}(T_{p,q}, a) \leq 1$ for every Alexander grading a.
- But this is not satisfied for $\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$.



K is *not* a torus knot.

Hyperbolic knot case

Pseudo-Anosov maps

- On the surface, there are two transverse foliations $\mathcal{F}^u, \mathcal{F}^s$, stretched by λ and compressed by $1/\lambda$.

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  - interior: k-prong for $k \geq 3$,
  - at a marked point: k-prong for $k \geq 1$,
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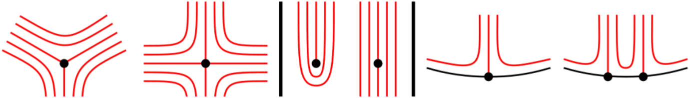


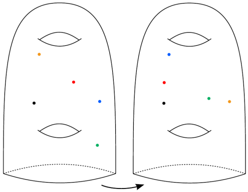
Counting fixed points from $\widehat{HFK}$

Theorem (Ghiggini–Spano, Ni)

If K is fibered hyperbolic with $\dim \widehat{HFK}(K, g-1) = r$, then the pseudo-Anosov monodromy is freely isotopic to one with at most $r-1$ interior fixed points.

For the granny: $\dim \widehat{HFK}(T_{2,3} \# T_{2,3}, 1) = 2$, so $r = 2$.

$\Rightarrow$ at most $\boxed{1}$ interior fixed point.



Fixed points: the black point is preserved by the map; the colored points permute.

Zero is impossible

Theorem (Farber–Reinoso–Wang)

There exist no hyperbolic knot in S^3 whose pseudo-Anosov monodromy on Σ_2^1 has zero interior fixed points.

$\Rightarrow$ h has exactly one interior fixed point, call it z.

Why we can cap off: boundary prong count ≥ 2

Goal: show the foliation of h has $k \geq 2$ prongs on $\partial\Sigma_2^1$. The free isotopy from h to pseudo-Anosov ϕ_h cyclically permutes the boundary prongs. The amount is measured by the fractional Dehn twist coefficient $c(h) =$k/n$\in \mathbb{Q}$.

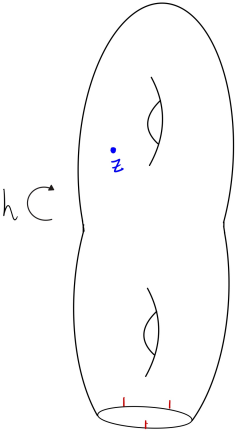

c(h) = 1/3

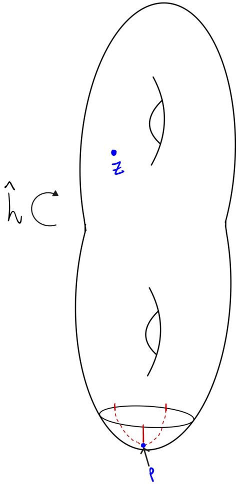
Two bounds on $c(h)$

- Lower (HKM, via quasipositive $\Rightarrow$ right-veering): $c(h) > 0$.
- Upper (Gabai-Oertel + HMK): $|c(h)| < 1$.

$$0 < c(h) < 1 \text{ non-integer} \quad \Rightarrow \quad \boxed{\text{prongs} \geq 2}.$$

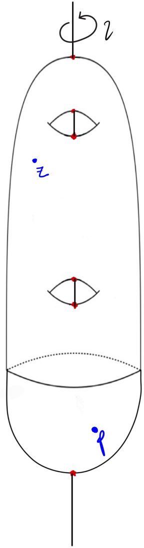
Capping off the boundary





The hyperelliptic involution

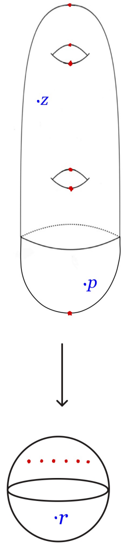
Σ_2^0 has an order-2 rotational symmetry ι, the hyperelliptic involution, with quotient S^2 branched over 6 points. Geometrically: rotation about a vertical axis piercing the surface at 6 points (the branch points of ι).



The hyperelliptic involution, quotient

Take the quotient under ι: cut, pinch, reshape $\Rightarrow$ a sphere with 6 marked points (images of the fixed points of ι). ι is central, so it

preserves $\mathrm{Fix}(\hat{h}) = \{z, p\}$. In fact ι swaps z and p, so they descend to a single point r.



Birman–Hilden descent

ι is central in $\mathrm{Mod}(\Sigma_2^0)$: every monodromy commutes with it.

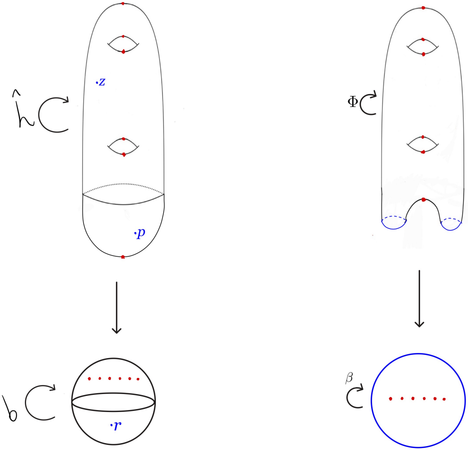
Theorem (Birman–Hilden)

In genus 2, every monodromy descends through the 2-to-1 cover to a mapping class on the 6-marked sphere:

$$\mathrm{Mod}(\Sigma_2^0)/\langle\iota\rangle \;\cong\; \mathrm{Mod}(S_{0,6}^2).$$

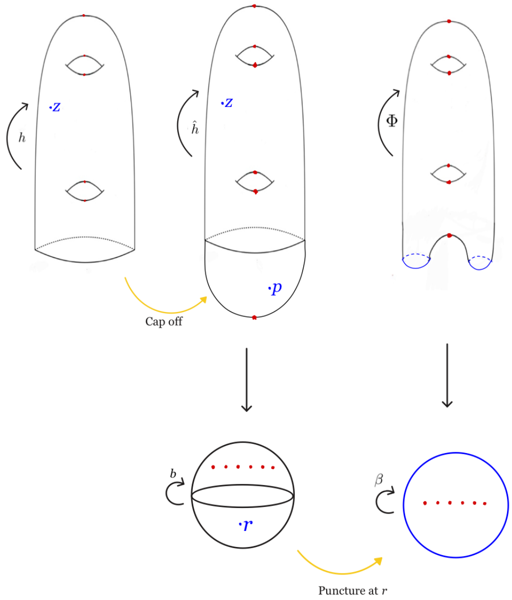
- Genus-2 monodromy $\rightsquigarrow$ braid on S^2 with 6 strands.
- Slightly easier surface to deal with.

Puncturing at r



Puncture at r to get a 6-braid β on D_6. Upstairs: puncture at z, p to get $\Phi : \Sigma_2^2 \to \Sigma_2^2$, the canonical lift. Φ has no interior fixed points.

The whole construction



Goal, restated

Classify all pseudo-Anosov mapping classes $\beta \in \mathrm{Mod}(D_6)$ lifting to fixed-point-free (FPF) maps on Σ_2^2, that could be induced by the monodromy of K.

Singularity types

Euler–Poincaré relates the number of prongs at singularities of a pseudo-Anosov map to the genus of the underlying surface. For us:

$$\sum_{\text{singularities}} \big(2 - \text{ prongs}\big) \;=\; 2\chi(\Sigma_2^0) \;=\; -4.$$

Where prongs ≥ 3. Solving:

$$\{3,3,3,3\} \quad \{3,3,4\} \quad \{4,4\} \quad \{5,3\} \quad \{6\}$$

Note: this is on the closed surface.

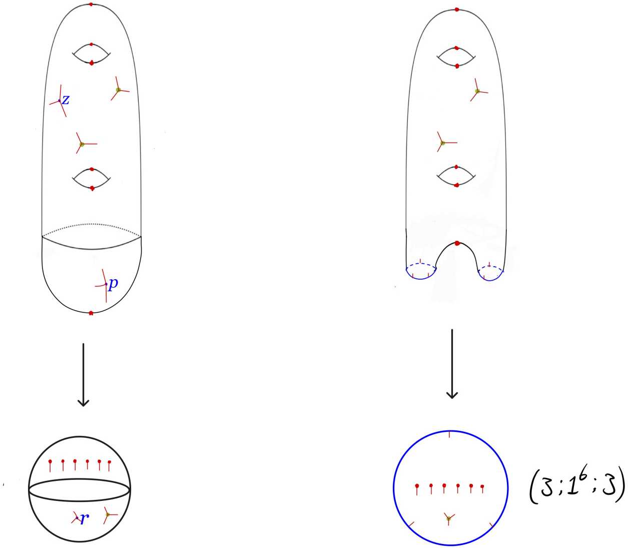
ι-symmetry kills three configurations

- ι swaps z and p, so they must have identical prong counts.
- Other singularities must permute in ι-orbits of equal prong count.
- Any prong-count-unique non-$\{z, p\}$ singularity would have to be a third fixed point — but $\mathrm{Fix}(\hat{h}) = \{z, p\}$.

Eliminates: $\{3, 3, 4\}, \{5, 3\}, \{6\}$. Survivors: $\{3, 3, 3, 3\}$ and $\{4, 4\}$.

Finitely many ways to configure these.

Prong configuration example $\{3, 3, 3, 3\}$



Five admissible strata on D_6

Tracking these through the Birman–Hilden construction:

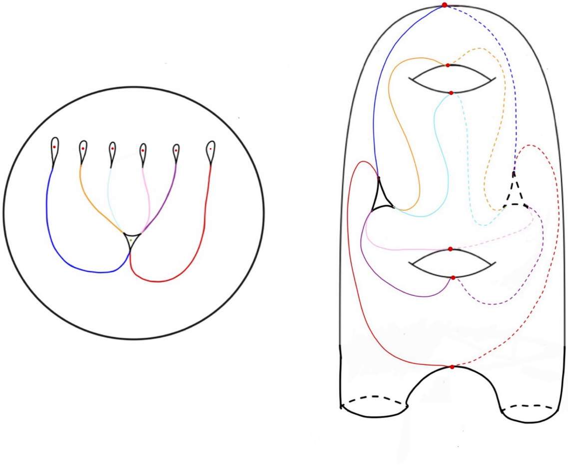
Stratum on D_6	**Type**
$(2; 1^4, 2^2; \varnothing)$	orientable
$(2; 1^6; 4)$	orientable
$(4; 1^6; \varnothing)$	orientable
$(3; 1^6; 3)$	non-orientable
$(2; 1^6; 3^2)$	non-orientable

Notation: $(b; m_1, \ldots; k_1, \ldots) =$ (boundary; marked-point; non-marked interior) prongs.

Goal, restated

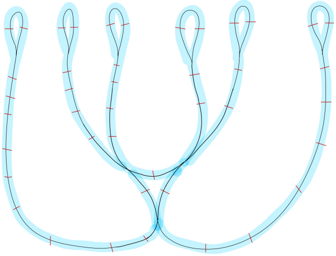
Classify all pseudo-Anosov mapping classes $\beta \in \mathrm{Mod}(D_6)$ on these strata lifting to fixed-point-free (FPF) maps on Σ_2^2, that could be induced by the monodromy of K.

Train tracks on surfaces



A train track is an embedded graph with smooth tangent structure at each vertex (switch).

Train tracks carry pseudo-Anosovs



Thicken the track → fibered neighborhood, foliated by vertical fibers.

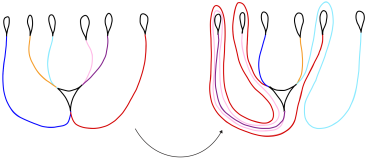
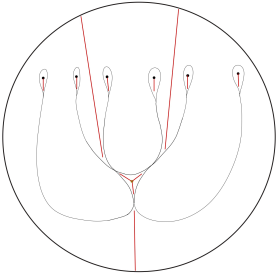


Image of τ runs transverse to the fibers → induced train track map f_τ.

We say τ carries the pseudo-Anosov.

Singularities and cusps



Singularities sit at the centers of complementary polygons.
k-prong singularity ⇔ *k*-sided polygon.
Peripherally: # cusps along ∂ = # boundary prongs.
Prongs of $\mathcal{F}^u$ emanate from singularity/boundary to cusps.

Why study train track maps?

Corollary 3.9 (Reinoso 2024, Cor. 2.7; after Bestvina–Handel)

Let $\alpha, \beta \in \mathrm{Mod}(D_n)$ be two pseudo-Anosov braids carried by the same train track τ, inducing the same train track map $f_\alpha = f_\beta : \tau \to \tau$. Then α and β are freely isotopic, and

$$\alpha = \Delta^{2k}\beta \text{ for some } k \in \mathbb{Z},$$

where $\Delta^2 = (\sigma_1 \cdots \sigma_{n-1})^n$ is the full twist about ∂D_n.

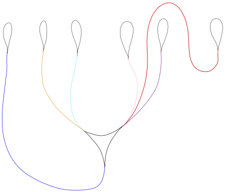
- Same train track map ⇒ same pseudo-Anosov representative.
- Underlying braids differ at most by a power of the full twist Δ^2.

Classifying pseudo-Anosov braids (up to adding full twists) reduces to classifying train track maps.

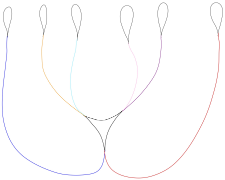
Standard and jointless

Theorem (Farber–Reinoso–Wang)

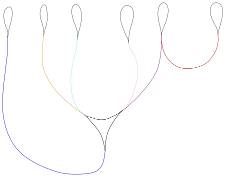
Any pseudo-Anosov braid is carried by a standard, jointless train track in the same stratum.



Non-standard



Standard and jointless



Standard with a joint

Standard: Only one edge above each marked point.
Jointless: Every marked-point polygon has no cusp on it.

Enumeration problem

A conundrum: Given a stratum, there are many such train tracks. Can we be sure we have all of them?

⇒ algorithmically enumerate all standard, jointless tracks in a stratum

Orientable lifts of the foliation

Three of the five strata — $(2; 1^4, 2^2; \varnothing)$, $(2; 1^6; 4)$, $(4; 1^6; \varnothing)$ — can be eliminated by an orientability argument.

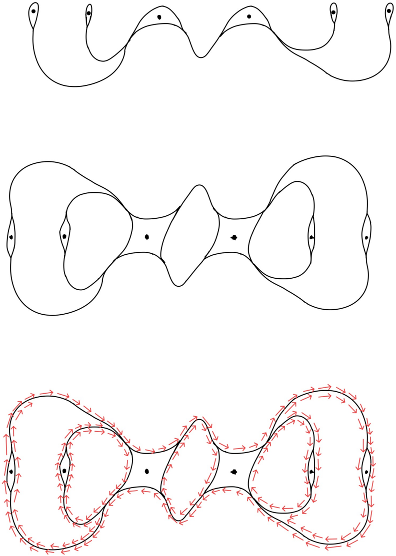
Orientability obstruction (Thurston)

If the lifted foliations on Σ_2^2 are orientable, then Φ acts on $H_1(\Sigma_2^2)$ with the dilatation λ as a real eigenvalue $\Rightarrow \lambda > 1$ is a real root of $\Delta_K(t)$.

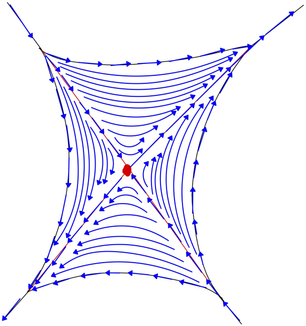
Orientable train tracks

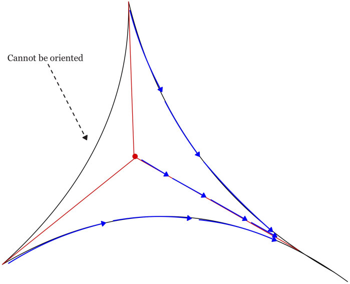
The lifted foliation is orientable in these strata because the train tracks in these strata are orientable.

Orientable train track in $(2; 1^4, 2^2; \varnothing)$



Necessary: Even-sided polygons





4-gon: orientation closes consistently. ✓

3-gon: parity fails. ×

Pick a direction for one leaf; orientations propagate around the polygon. Even sides closes; odd sides forces an inconsistency.

Three down, two to go

$(t^2 - t + 1)^2$ has no real roots — its roots are primitive 6th roots of unity.

No real $\lambda > 1$. Contradiction.

The three orientable strata are eliminated.

Eliminated: $(2; 1^4, 2^2; \varnothing), (2; 1^6; 4), (4; 1^6; \varnothing)$
Remaining: $(3; 1^6; 3)$ and $(2; 1^6; 3^2)$. Both are non-orientable.

Goal, restated

Classify all train track maps on standard jointless train tracks
on the two non-orientable strata
induced by $\beta \in \mathrm{Mod}(D_6)$ lifting to fixed-point-free (FPF)
maps on Σ_2^2.

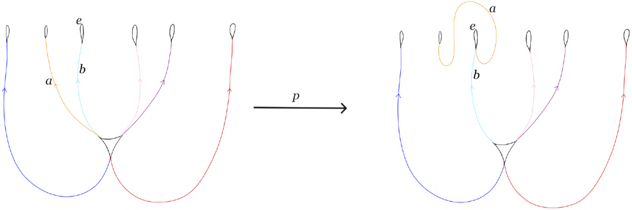
Enumerating train tracks

- We need to enumerate all standard jointless tracks on D_6 in the strata $(3; 1^6; 3)$ and $(2; 1^6; 3^2)$.
- There might be many. Have we gotten them all?
- Idea: Use folding automaton to algorithmically enumerate all standard train tracks, then filter by jointless.

Elementary folding

Theorem (Bestvina–Handel 1995)

Every train track map decomposes as a finite sequence of elementary folds.



At a cusp (a, b) at a switch, fold a onto b. In real-edge words: $p(a) = b \cdot a'$.

The folding automaton

Original idea (Ko–Los–Song 2002; Los 2008)

From a starting track τ, perform all possible folds; check if each result is new; repeat. This process terminates.

`autofolder` (Y. 2026)

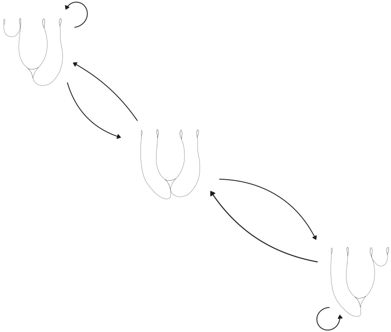
An explicit algorithmic realization of the Ko–Los–Song folding automaton.

Theorem (Y.; implementing Ko–Los–Song / Ham–Song)

For any stratum on D_n, the folding automaton has finitely many vertices and is connected as an undirected graph. Every pseudo-Anosov mapping class in the stratum is carried by some vertex.

The folding automaton: example

Vertices: standard tracks Edges: elementary folds. Every pseudo-Anosov in the stratum is a cycle in the automaton (Ko–Los–Song).



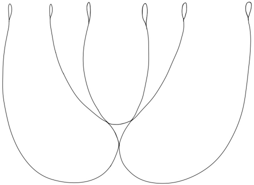
Folding automaton of stratum $(1; 1^4; 3)$ on D_4.

`autofolder` in practice

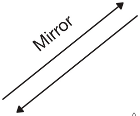
Python on SageMath. Input: a stratum. Output: the full folding automaton.

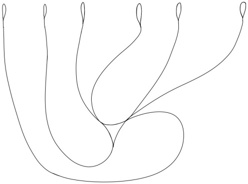
Stratum	**Standard**	**Jointless**
$(3; 1^6; 3)$	138	4
$(2; 1^6; 3^2)$	110	20

`autofolder` output, standard jointless tracks on $(3; 1^6; 3)$



(0.) The candelabra





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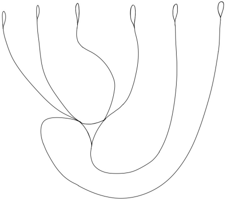

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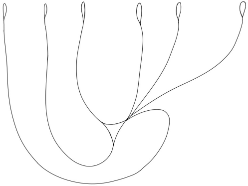
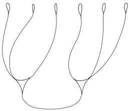


(3.)

`autofolder` output, standard jointless tracks on $(2^2; 1^6; 3^2)$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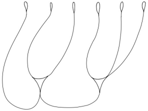
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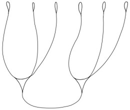
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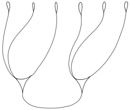
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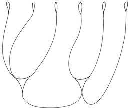
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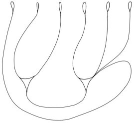
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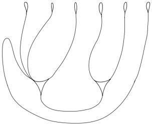
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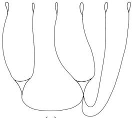
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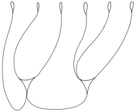
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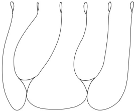
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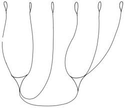
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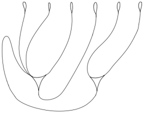
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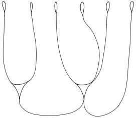
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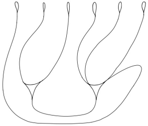
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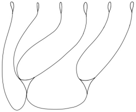
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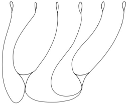
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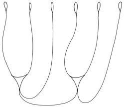
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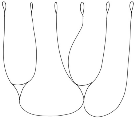
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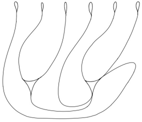
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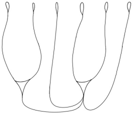
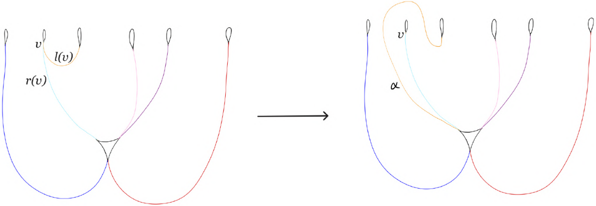

(19.)

`autofolder` output: standard, jointless, modulo mirrors

Stratum	**Standard**	**Jointless**	**Mirror**
$(3;1^6;3)$	138	4	3
$(2;1^6;3^2)$	110	20	12

Tight splitting

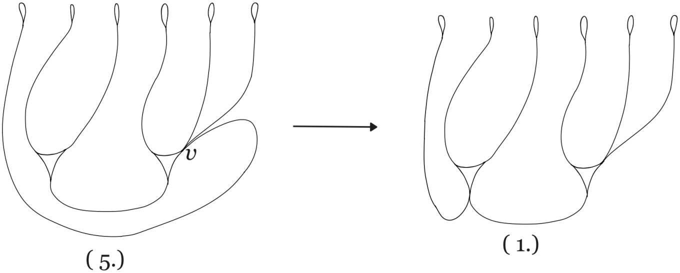
A split at a switch v is the inverse of a fold.



Theorem (Farber–Reinoso–Wang)

A switch of maximum real valence > 1 at a polygon can always be split so that the resulting track still carries the same map the original track does.

Splitting reduces the candidate list



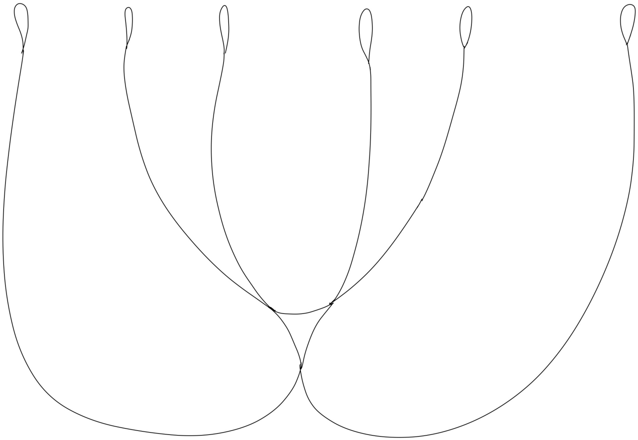
Every train track map carried by 5 is carried by 1.

Reduction: Jointless+Mirror+Splitting

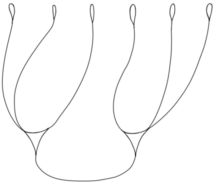
Stratum	**Standard**	**Jointless**	**Mirror**	**Splitting**
$(3; 1^6; 3)$	138	4	3	1
$(2; 1^6; 3^2)$	110	20	12	6

We are now ready to begin train track map analysis on these tracks.

$(3; 1^6; 3)$, the Candelabra



$(2; 1^6; 3^2)$, six train tracks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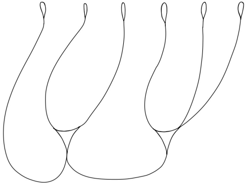
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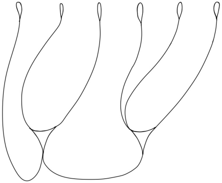
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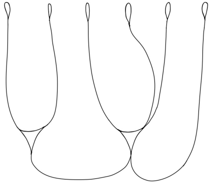
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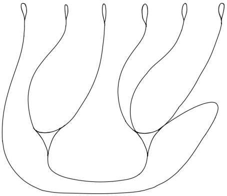
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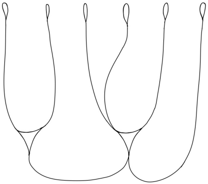

(17.)

Goal, restated once more

Classify all train track maps on the seven tracks above such that the pseudo-Anosov braids inducing them, lift to FPF maps on Σ_2^2.

The Trace Lemma

Train track maps on D_6 lift via Birman–Hilden to Σ_2^2. The Trace Lemma constrains which lifts can be FPF upstairs.

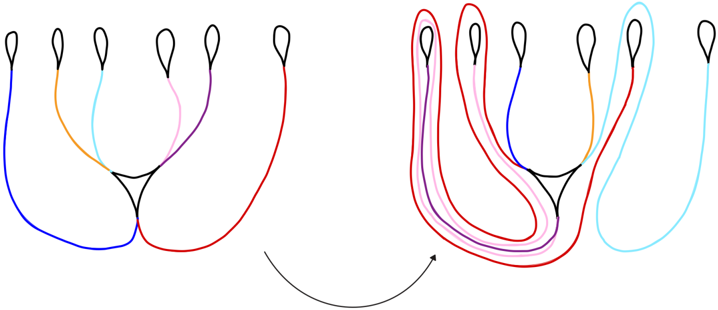
Trace Lemma (original; via Lefschetz)

If the lifted pseudo-Anosov Φ on Σ_2^2 is FPF, then its transition matrix is traceless.

Trace Lemma, jointless version downstairs

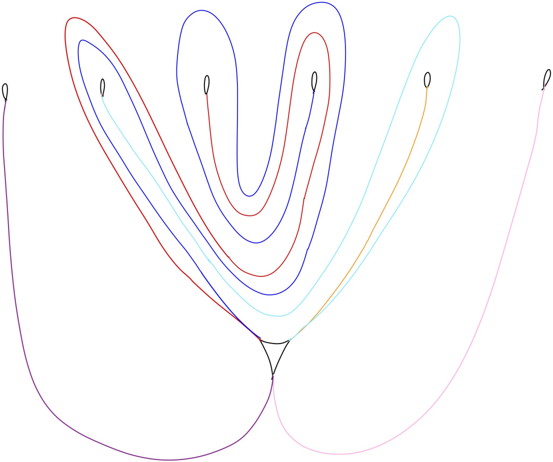
Downstairs: if a real edge e is incident to a jointless monogon, then e does not appear in $f_\beta(e)$.

Trace Lemma in Practice

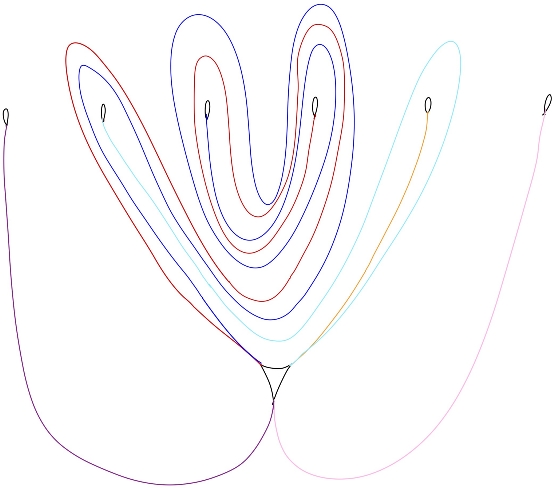


The image of every real edge does not trace itself.

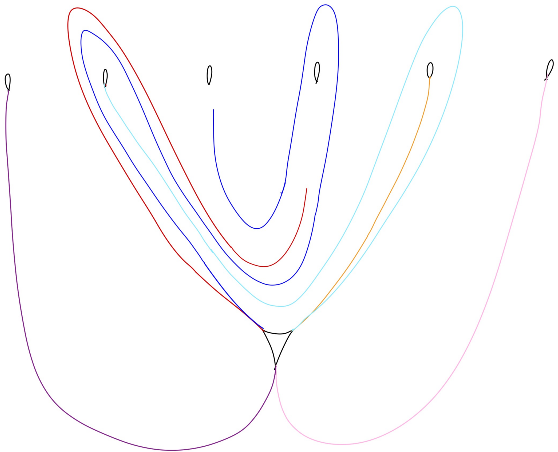
Case analysis example



Case analysis example



Case analysis example



Candidate families after case analysis

Stratum	**Track**	**Families**
$(3;1^6;3)$	Candelabra	38
$(2;1^6;3^2)$	Track 0	13
	Track 1	17
	Track 8	9
	Track 12	0
	Track 13	4
	Track 17	6
	subtotal	49
Total		87

Each family is a 1-, 2-, or 3-parameter braid word in $\mathrm{Mod}(D_6)$.

Goal, restated — in terms of K

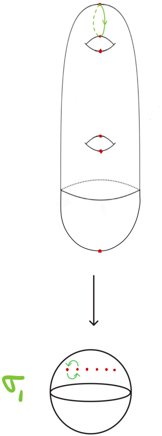
Theorem (Y.)

Every FPF pseudo-Anosov on Σ_2^2 in either non-orientable stratum is conjugate to the lift of one of 87 explicit braid families, up to composition with full twists and mirroring.

For each of the 87 candidate braid families β,
determine whether β can be induced by (via Birman–Hilden) the monodromy of a knot $K \subset S^3$
with $\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$.

Two algebraic filters: $K \subset S^3$ (*F1*) and $\Delta_K = (t^2 - t + 1)^2$ (*F2*).

Homological filtering



- Each braid generator is a Dehn twist upstairs, which on the generators of $H_1(\Sigma_2;\mathbb{Z})$, representable as a matrix. Composing:

$$(\hat{h})_* \ : \ H_1(\Sigma_2;\mathbb{Z}) \to H_1(\Sigma_2;\mathbb{Z}),$$

  a 4 × 4 integer matrix.

Two algebraic tests follow.

Filter F1: zero-surgery condition

For K to be a knot in S^3, the 0-surgery $S^3_0(K)$ must have

$$H_1(S^3_0(K);\mathbb{Z}) \;\cong\; \mathbb{Z}.$$

Computing H_1 from $\hat{h}_*$ on $H_1(\Sigma^0_2)$, this becomes:

$$\boxed{\;\left|\det(I - (\hat{h})_*)\right| \;=\; 1\;}$$

Most candidate families fail F1 immediately — determinant comes out ≥ 2.

Filter F2: Alexander polynomial

The characteristic polynomial of $(\hat{h})_*$ must match the granny's Alexander polynomial:

$$\boxed{\det(tI - (\hat{h})_*) \;=\; (t^2 - t + 1)^2}$$

- This eliminates everything F1 didn't.
- Some families pass F1 but fail F2 (or vice versa) at every integer parameter value.

Verdict

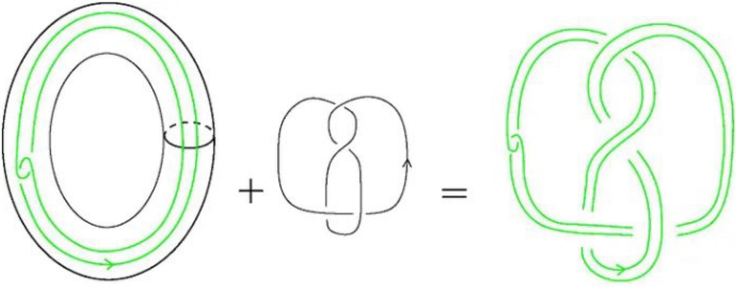
Of the $\boxed{87}$ candidate families,

zero pass both F1 and F2.

The hyperbolic case is fully eliminated.

Satellite knot case

Satellite knots, briefly



$P \subset S^1 \times D^2$ pattern knot

$C \subset S^3$ companion knot

$K = P(C) \subset S^3$ satellite knot

Schubert's formula relates Δ_K to the Alexander polynomials of P, C.

Schubert's formula

Theorem (Schubert, 1953)

If $K = P(C)$ is a satellite with pattern P, companion C, and winding number w, then

$$\Delta_K(t) \doteq \Delta_{P(U)}(t) \cdot \Delta_C(t^w).$$

Plug in $\Delta_K(t) = (t^2 - t + 1)^2$ and impose fiberedness:

$$P(U) \text{ is a trefoil}, \quad C \text{ is a trefoil}, \quad w = 1.$$

(Symmetric factorization; the alternative $\Delta_{P(U)} = 1$ forces $K = C$ trivially.)

Thurston reducing system

- Thurston reducing system $\Gamma = \gamma_1, \ldots, \gamma_n$ is set of multi-curves preserved by the map.
- Map fixes outer piece $h_0 \colon \Sigma_0 \to \Sigma_0$, and is either periodic or pseudo-Anosov.
- Three configurations on Σ_2^1.

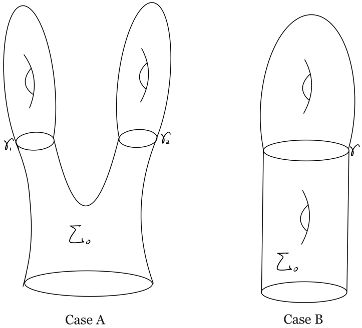


Figure 8.1: the two non-trivial cases of Γ*.*

Case A: Σ_0 pair of pants

- Pair of pants has no pseudo-Anosov homeomorphisms.
  $\Rightarrow$ outer map h_0 is periodic.
- Baldwin–Sivek (Proposition 2.3 of [BS25]): for periodic outer maps with $\Gamma \neq \emptyset$, K is one of:
  - a torus knot (killed),
  - a (p,q)-cable with Σ_0 of genus $g(T_{p,q}) = 0$ — forces $C =$ unknot, contradiction,
  - composite, with Σ_0 planar.
- Only composite genus-2 fibered knot with $\Delta = (t^2 - t + 1)^2$:

connect-sum of two trefoils.

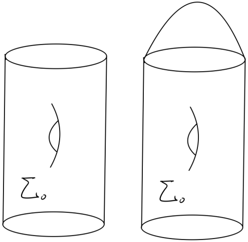
Case B: Σ_0 is genus-1

Outer map ϕ_0 is periodic or pseudo-Anosov.

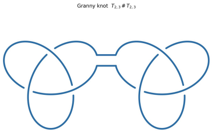
Periodic $\Rightarrow$ (p, q)-cable. Pattern is the trefoil. But $T_{2,3}$ as a cable has winding number 2 or 3, contradicting $w = 1$.

Pseudo-Anosov $\Rightarrow$ capping off γ recovers the trefoil monodromy — not pA. Forces $|c(h)| = 1$ exactly. A contact-invariant argument rules this out.

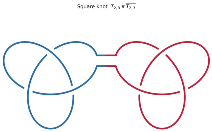
Case B is empty.



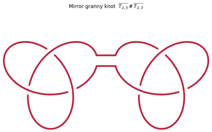
Three candidates remain



Granny $T_{2,3}\#T_{2,3}$



Square $T_{2,3}\#\overline{T}_{2,3}$



Mirror granny $\overline{T}_{2,3}\#\overline{T}_{2,3}$

The trichotomy has narrowed K to a connect-sum of two trefoils.
Three options remain.

The bigrading separates them

The three knots share the same Alexander polynomial and the same total $\widehat{HFK}$-rank, but their bigraded $\widehat{HFK}$'s are pairwise distinct.

Only the granny matches $\widehat{HFK}(K)$.

Main Theorem, proved

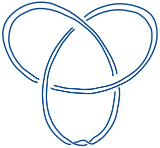
Among the three candidates, only the granny matches $\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$ as bigraded vector spaces.

Main Theorem: $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$. ■

Three corollaries

- First $\widehat{HFK}$-detection of a composite knot.
- Combined with Binns: $\widehat{HFK}$ detects the unique composite almost L-space knot (Corollary 1.8).

A second target



$C_{2,1}(T_{2,3})$: the $(2,1)$-cable of the right-handed trefoil. Genus 2, fibered. Monodromy also one fixed point. Alexander polynomial:

$$\Delta_{C_{2,1}(T_{2,3})}(t) \;=\; t^4 - t^2 + 1 \;=\; \Phi_{12}(t).$$

- Different from $(t^2 - t + 1)^2$, so not a Main Theorem candidate.
- Same kind of object as granny — fibered genus-2 with one fixed point in the monodromy — same machinery applies.

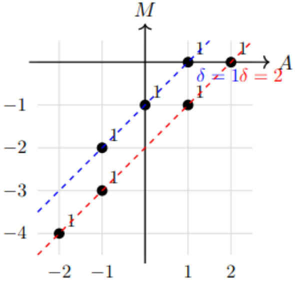
Theorem 1.6: partial detection of the cable

Theorem 1.6

If $\widehat{HFK}(K) \cong \widehat{HFK}(C_{2,1}(T_{2,3}))$ as bigraded $\mathbb{F}_2$-vector spaces, then either

- $K \cong C_{2,1}(T_{2,3})$, or
- K is hyperbolic, with disk braid in one of 6 explicitly enumerated parametric families.

Partial detection: residual list not closed by homological filters alone.



$\widehat{HFK}(C_{2,1}(T_{2,3}); \mathbb{F}_2)$

Corollary 1.7

Corollary (Y.)

$\widehat{HFK}$ detects $C_{2,1}(T_{2,3})$ among *non-hyperbolic* knots.

Six future directions *(for young grad students)*

1. Close the cable detection gap.
2. Genus ≥ 3 targets (the hyperelliptic constraint).
3. Targets with multiple interior fixed points (next-to-top rank ≥ 3).
4. Khovanov detection of the granny.
5. The Hedden–Watson pretzel family and HFK/Khovanov complementarity.
6. `autofolder`: Enumerate pseudo-Anosovs with bounded dilatation.

Closing the cable gap

Two strategies for the 6 residual families:

(A) Check Pseudo-Anosov-ness. Write down and check transition matrix.

(B) Check FPF. Cotton-Clay formula.

(C) Direct knot reconstruction via SnapPy.

This would give first $\widehat{HFK}$ detection of a prime satellite.

Genus ≥ 3: the hyperelliptic constraint

In genus $g \geq 3$, the hyperelliptic mapping class group

$$H_g \;=\; C_{\mathrm{Mod}(\Sigma_g)}(\iota)$$

is a proper subgroup. The framework extends only to *hyperelliptic* monodromies.

If it is known a fibered knot has hyperelliptic monodromy, the strategy can be used.

Beyond two fixed points

What if the monodromy has more than one fixed point? Extend the fact that ι swaps the fixed point and the additional one from capping off:

Lemma (Y.)

For a fibered knot in an integer homology sphere with hyperelliptic-symmetric capped monodromy, ι pairs the fixed points of $\hat{h}$.

Concrete genus-2 hyperbolic fibered targets in S^3 with rank 3: 10_{132}, 10_{145}, $K11n38$.

Train track analysis might be very complicated. Automate?

Khovanov detection of the granny

Khovanov homology detects: unknot, trefoils, figure-eight, $T_{2,5}, T_{-2,5}, 5_2$, certain Whitehead doubles — all prime. **Proposed strategy** for the granny (three steps):

1. Bigraded *Kh*-iso ⇒ K is Khovanov-thin.
2. Dowlin's spectral sequence ⇒ $\widehat{HFK}(K)$ thin ⇒ $\widehat{HFK}(K;\mathbb{Q}) \cong \widehat{HFK}(T_{2,3} \# T_{2,3};\mathbb{Q})$.
3. Extend Theorem 1.1 from $\mathbb{F}_2$ to $\mathbb{Q}$ coefficients.

Would be the first Khovanov detection of a composite knot.

Why the square knot is genuinely harder

Hedden–Watson + Wang: the pretzel family $\{P(3,-3,2n)\}_{n\in\mathbb{Z}}$ contains the square knot at $n=0$, with identical $\widehat{HFK}$ and instanton $\widehat{HFK}$ for every member. $\Rightarrow$ $\widehat{HFK}$ cannot detect the square knot.

A conjectural complementarity

Question. Does $\widehat{HFK}$ detect membership in the pretzel family $\{P(3,-3,2n)\}$? If so:

- $\widehat{HFK}$ would identify the family (modulo HFK-equivalence),
- Khovanov would resolve the specific member.

First known invariant pair coarsely detecting a non-trivial infinite family.

`autofolder` beyond detection

In 2008, Los called for an explicit algorithmic realization of the folding automaton, noting that without one, the state space remains intractable.

`autofolder` answers that call.

Standalone applications of the algorithm:

- Dilatation enumeration via Perron–Frobenius bounds on transition matrices.
- Pseudo-Anosov classification within strata.
- Edge data (standardizing braids, transition matrices) is computed but not yet exposed — activating it enables finer dynamical invariants.
- Closed-surface generalization: would enable detection of fibered knots in 3-manifolds beyond S^3.

Available as a Python / SageMath package.

Open problem, cable detection

Open problem (Yeh, 2026)

Prove that $\widehat{HFK}$ detects the $(2,1)$-cable of the trefoil, or find a counterexample to show it does not.

Thank you

Knot Floer homology detects the granny knot.

- Advisor: John A. Baldwin.
- Intellectual co-author of the framework: Braeden Reinoso.
- Committee, collaborators, family.

How the story ended.

Backup slides

Why ι swaps z and p — detailed

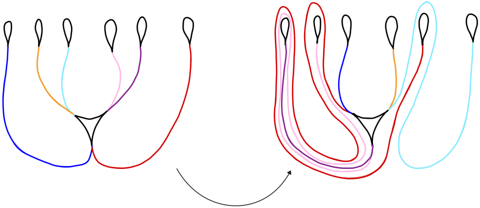
Suppose ι fixed both z and p. Then both become Weierstrass points; descending, h would lift a 5-braid β on D_5. Since $\hat{\phi}_h$ fixes z and z descends to a branch point, the braid permutation σ_β has a fixed index $\Rightarrow$ closure $\hat{\beta}$ has $\mu(\hat{\beta}) \geq 2$ link components. The mapping torus M_h is the branched double cover of S^3 along $\hat{\beta}$:

$$M_h \;\cong\; \Sigma\big(S^3, \hat{\beta}\big).$$

For a μ-component link, $b_1\big(\Sigma(S^3, \hat{\beta})\big) \geq \mu - 1 \geq 1$. But $K \subset S^3$ a knot $\Rightarrow$ $M_h \cong S^3 \Rightarrow b_1(M_h) = 0$.

Contradiction. So ι swaps z and p.

Transition matrix



	blue	orange	sky blue	pink	purple	red
blue	0	0	1	0	0	0
orange	0	0	0	0	1	0
sky blue	0	0	0	0	2	1
pink	2	1	0	0	0	0
purple	1	0	0	0	0	0
red	2	2	1	0	1	0

Transition matrices count traversals of real edges only.

Case analysis example