Knot Floer Homology, Folding Automata, and the Connect-Sum of Trefoils

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Two years ago, at BC...

Dr. Braeden Reinoso closed his defense by posing:

Open problem (Reinoso, 2024)

Prove that $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$ (the granny knot), or find a counterexample to show it does not.

Today: how this story ended.

Main result

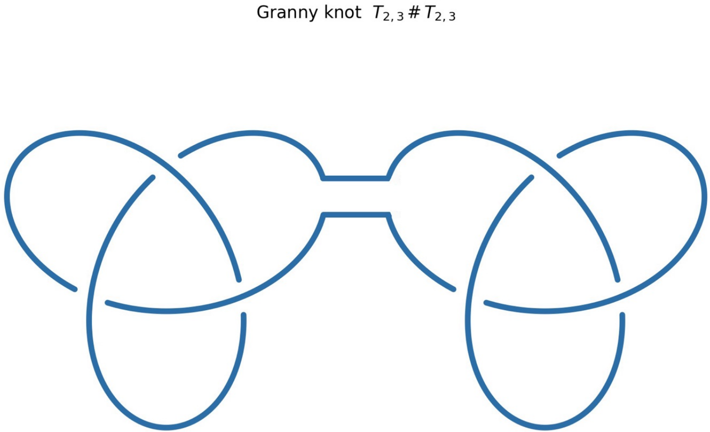
Main Theorem

Knot Floer homology ($\widehat{HFK}$) detects $T_{2,3} \# T_{2,3}$.

- First detection result in $\widehat{HFK}$ of a composite knot.

Building on Ozsváth–Szabó, Ni, Baldwin–Hu–Sivek, Farber–Reinoso–Wang, Reinoso

The granny knot, $T_{2,3} \# T_{2,3}$



Knot invariants and detection

- Knot invariant — a rule assigning to each knot K a mathematical, often algebraic object (polynomial, group, vector space, etc.).

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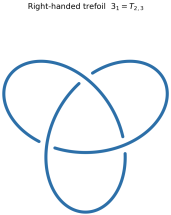
- Knot Floer homology $\widehat{HFK}$ is a knot invariant. The output is a bigraded vector space.
- Detection is rare! "Classical" invariants not known to detect even the unknot.
- Knot Floer homology is known to detect: unknot, trefoils, figure-eight, $T_{2,5}$, $T_{-2,5}$ ...

Prime vs composite knots

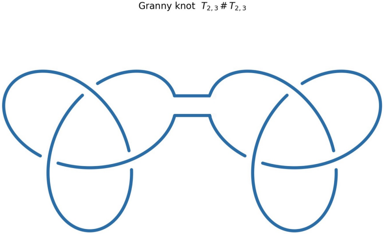
A knot is composite if it splits as a *connect-sum* $K = K_1 \# K_2$ of two non-trivial knots. Otherwise it is prime.

Prime



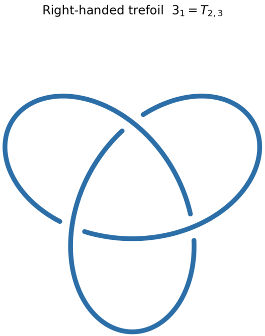
right-handed trefoil $T_{2,3}$

Composite



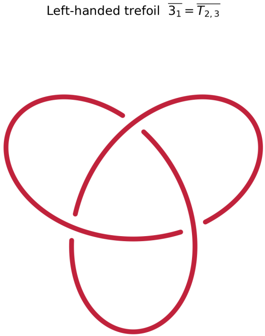
connect-sum $T_{2,3} \# T_{2,3}$, the granny knot

Two trefoils



right-handed trefoil, $T_{2,3}$

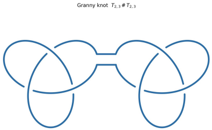
right-handed crossing



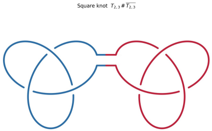
left-handed trefoil $\overline{T_{2,3}}$

left-handed crossing

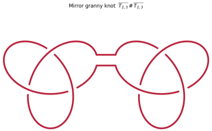
Three connect-sums of two trefoils



Granny $T_{2,3} \# T_{2,3}$



Square $T_{2,3} \# \overline{T}_{2,3}$



Mirror granny $\overline{T}_{2,3} \# \overline{T}_{2,3}$

All three share $\Delta_K(t) = (t^2 - t + 1)^2$.

Knot Floer homology (Ozsvath–Szabo, Rasmussen, around 2003)

- $\widehat{HFK}(K; \mathbb{F}_2)$ is a bigraded $\mathbb{F}_2$-vector space, with Alexander grading A and Maslov grading M.

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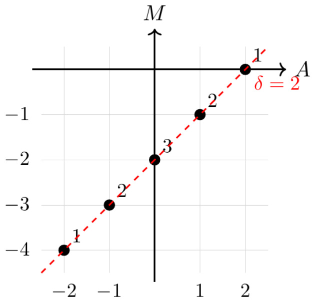
The granny knot:

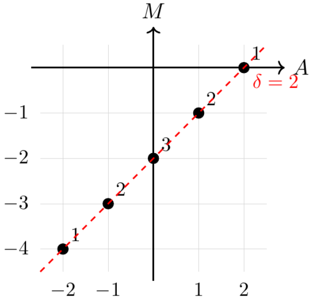


Main result, formally

Main Theorem

If $\widehat{HFK}(K;\mathbb{F}_2) \cong \widehat{HFK}(T_{2,3}\#T_{2,3};\mathbb{F}_2)$ as bigraded vector spaces, then $K \cong T_{2,3}\#T_{2,3}$.

- Feed an arbitrary knot K into the blackbox $\widehat{HFK}$.
- If the output is exactly this bigraded grid *(right)* $\Rightarrow$ K was the granny knot.



$\widehat{HFK}(T_{2,3}\#T_{2,3};\mathbb{F}_2)$

$\widehat{HFK}$ detects fiberedness

Theorem (Ozsváth–Szabó; Ni; Ghiggini; Juhász)

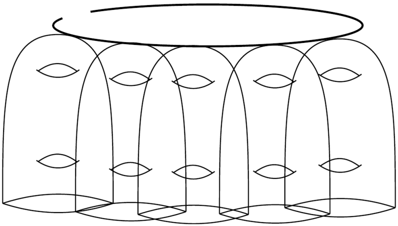
$\widehat{HFK}$ detects both Seifert genus ($g(K) = \max\{A : \widehat{HFK}(K, A) \neq 0\}$) and fiberedness ($K$ fibered $\iff \dim \widehat{HFK}(K, g(K)) = 1$).

Hence: K with HFK matching $T_{2,3} \# T_{2,3}$ is fibered of genus 2.

What does fibered mean?

A knot is *fibered* if its exterior is foliated by an S^1-family of Seifert surfaces. Equivalently: K

is the binding of an *open book*, with pages a genus-g Seifert surface fanned around the binding.



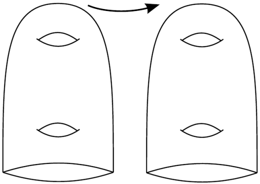
Fibered knot → surface map

As the Seifert surface sweeps around, it foliates the knot exterior — and after one full turn returns to itself via a homeomorphism

$$h : \Sigma \to \Sigma, \quad h|_{\partial\Sigma} = \mathrm{id}.$$

h is the monodromy. The

pair (Σ, h) determines K up to isotopy.



⇒ *reduce the problem:* knowing K is fibered of genus-2, study the monodromy h, argue it can only be the monodromy of $T_{2,3} \# T_{2,3}$.

Mapping class group of a surface

- h is well-defined up to isotopy rel boundary and conjugation. The natural setting:

$$\mathrm{Mod}(\Sigma) \;=\; \mathrm{Homeo}^{+}(\Sigma)\,/\,\text{isotopy}.$$

- Our monodromy: $[h] \in \mathrm{Mod}(\Sigma_2^1)$.
- Restated goal: instead of asking "which knot has the granny's HFK?", we now ask: "up to conjugation, which mapping class on the genus-two surface could be the monodromy of K?" Same question.

Theorem (Thurston)

Every knot K in S^3 is exactly one of the following:

- Torus knot
- Hyperbolic knot
- Satellite knot

For a fibered knot K, in each case the monodromy h is freely isotopic to a representative ϕ_h:

Theorem (Nielsen–Thurston)

- K is a torus knot $\Rightarrow \phi_h$ is *periodic* (some power is identity);
- K is hyperbolic $\Rightarrow \phi_h$ is *pseudo-Anosov* (Stretches and compresses a pair of foliations);
- K is a satellite $\Rightarrow \phi_h$ is *reducible* (fixes a set of curves).

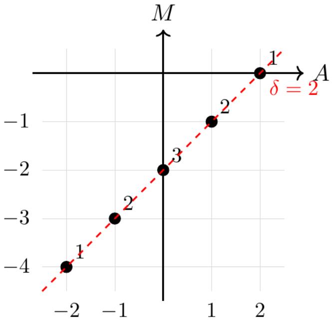
Three cases for our fibered genus-2 knot K:

Knot class	Monodromy	Outcome
Torus	periodic	ruled out by HFK rank
Hyperbolic	pseudo-Anosov	most of the talk, ruled out by train tracks
Satellite	reducible	forces connect-sum of trefoils

Torus knot case

K is not a torus knot

- Torus knots are known to satisfy $\dim \widehat{HFK}(T_{p,q}, a) \leq 1$ for every Alexander grading a.
- But this is not satisfied for $\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$.



K is *not* a torus knot.

Hyperbolic knot case

Pseudo-Anosov maps

- On the (possibly marked) surface, there are two transverse foliations $\mathcal{F}^u, \mathcal{F}^s$, stretched by λ and compressed by $1/\lambda$.

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- Foliations have singularities with prongs. Allowable local models:
  - interior: k-prong for $k \geq 3$,
  - at a marked point: k-prong for $k \geq 1$,
  - on a boundary component: k-prong for $k \geq 1$.

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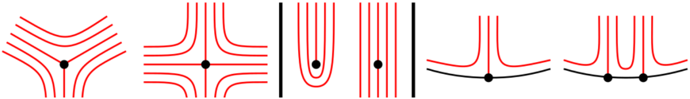
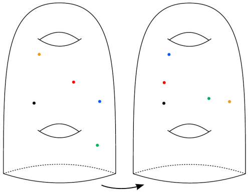


Counting fixed points from $\widehat{HFK}$

Theorem (Ghiggini–Spano, Ni)

If K is fibered hyperbolic with $\dim \widehat{HFK}(K, g-1) = r$, then the pseudo-Anosov monodromy is freely isotopic to one with at most $r-1$ interior fixed points.



Fixed points: the black point is preserved by the map; the colored points permute.

For the granny: $\dim \widehat{HFK}(T_{2,3} \# T_{2,3}, 1) = 2$, so $r = 2$.

$\Rightarrow$ at most $\boxed{1}$ interior fixed point.

Zero is impossible

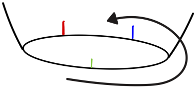
Theorem (Farber–Reinoso–Wang)

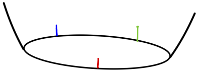
There exist no hyperbolic knot in S^3 whose pseudo-Anosov monodromy on Σ_2^1 has zero interior fixed points.

$\Rightarrow \phi_h$ has exactly one interior fixed point, call it z.

Why we can cap off: boundary prong count ≥ 2

Goal: show the foliation of h has $k \geq 2$ prongs on $\partial \Sigma_2^1$. The free isotopy from h to pseudo-Anosov ϕ_h cyclically permutes the boundary prongs. The amount is measured by the fractional Dehn twist coefficient $c(h) = \frac{k}{(\#\text{prongs})} \in \mathbb{Q}$.





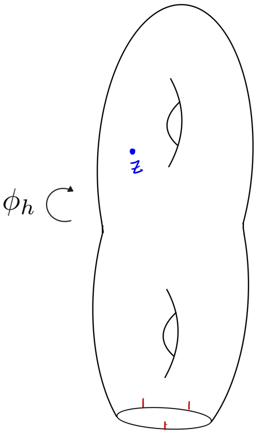
$c(h) = 1/3$

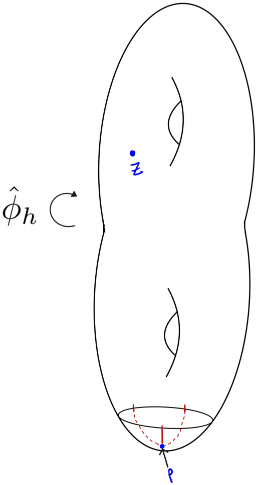
Two bounds on $c(h)$

- Lower (HKM, via strongly quasipositivity $\Rightarrow$ right-veering): $c(h) > 0$.
- Upper (Gabai-Oertel): $|c(h)| < 1$.

$$0 < c(h) < 1 \text{ non-integer} \quad \Rightarrow \quad \boxed{\text{prongs} \geq 2}.$$

Capping off the boundary

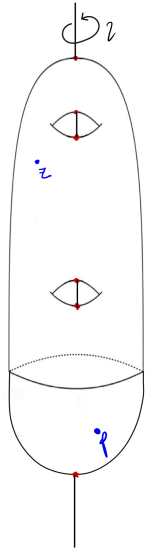




The hyperelliptic involution

Σ_2^0 has an order-2 rotational symmetry ι, the hyperelliptic involution, with quotient S^2 branched over 6 points. Geometrically: rotation about a

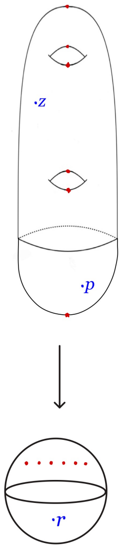
vertical axis piercing the surface at 6 points (the branch points of ι).



The hyperelliptic involution, quotient

Take the quotient under ι: cut, pinch, reshape $\Rightarrow$ a sphere with 6 marked points (images of the fixed points of ι). ι is central, so it

preserves $\mathrm{Fix}(\hat{h}) = \{z, p\}$. In fact ι swaps z and p, so they descend to a single point r.



Birman–Hilden descent

ι is central in $\mathrm{Mod}(\Sigma_2^0)$: every monodromy commutes with it.

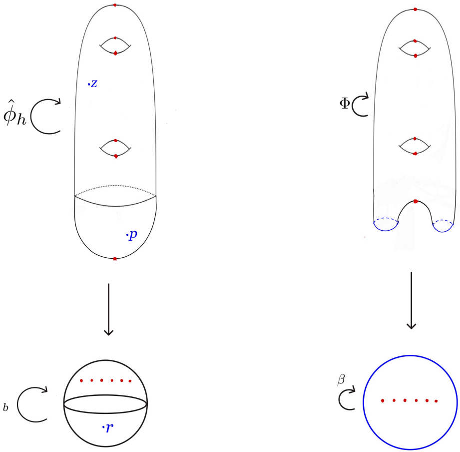
Theorem (Birman–Hilden)

In genus 2, every monodromy descends through the 2-to-1 cover to a mapping class on the 6-marked sphere:

$$\mathrm{Mod}(\Sigma_2^0)/\langle\iota\rangle \;\cong\; \mathrm{Mod}(S_{0,6}^2).$$

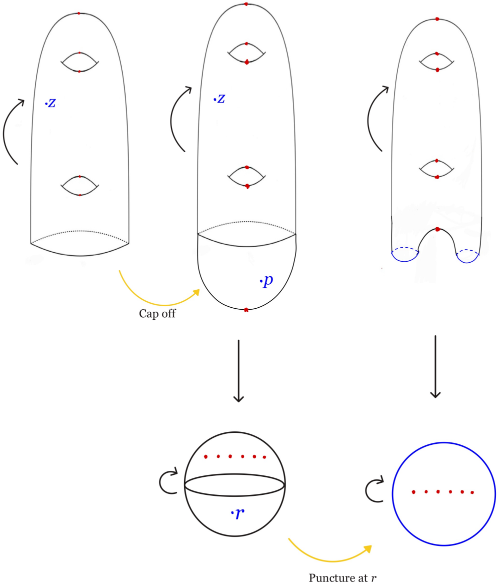
- Genus-2 monodromy $\rightsquigarrow$ braid on S^2 with 6 strands.
- Slightly easier surface to deal with.

Puncturing at r



Puncture at r to get a 6-braid β on D_6. Upstairs: puncture at z, p to get $\Phi : \Sigma_2^2 \to \Sigma_2^2$, the canonical lift. Φ has no interior fixed points.

The whole Birman-Hilden construction



Goal, restated

Classify all pseudo-Anosov maps on D_6 lifting to fixed-point-free (FPF) maps on Σ_2^2, that could be induced by the monodromy of K.

Singularity types

Euler–Poincaré relates the number of prongs at singularities of a pseudo-Anosov map to the genus of the underlying surface. For us:

$$\sum_{\text{singularities}} \big(2 - \text{ prongs}\big) \;=\; 2\chi(\Sigma_2^0) \;=\; -4.$$

Where prongs ≥ 3. Solving:

$$\{3,3,3,3\} \quad \{3,3,4\} \quad \{4,4\} \quad \{5,3\} \quad \{6\}$$

Note: this is on the closed surface.

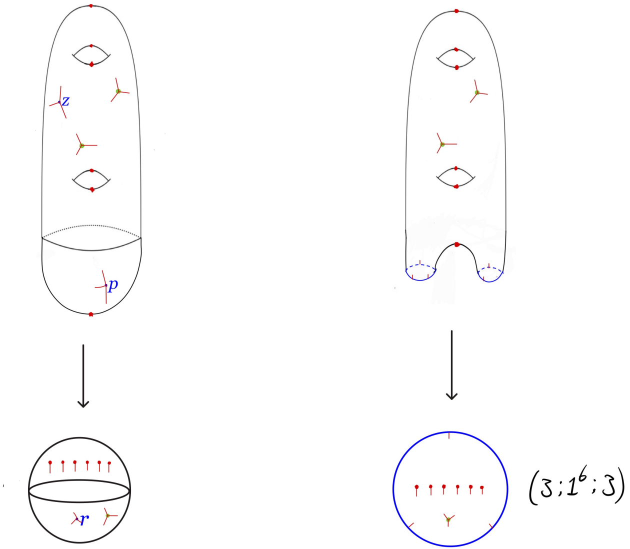
ι-symmetry kills three configurations

- ι swaps z and p, so they must have identical prong counts.
- Other singularities must permute in ι-orbits of equal prong count.
- Any prong-count-unique non-$\{z, p\}$ singularity would have to be a third fixed point — but $\mathrm{Fix}(\hat{h}) = \{z, p\}$.

Eliminates: $\{3, 3, 4\}, \{5, 3\}, \{6\}$. Survivors: $\{3, 3, 3, 3\}$ and $\{4, 4\}$.

Finitely many ways to configure these.

Prong configuration example $\{3, 3, 3, 3\}$



Five admissible strata on D_6

Tracking these through the Birman–Hilden construction:

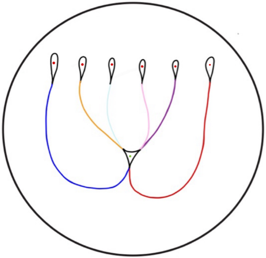
Stratum on D_6
$(2; 1^4, 2^2; \varnothing)$
$(2; 1^6; 4)$
$(4; 1^6; \varnothing)$
$(3; 1^6; 3)$
$(2; 1^6; 3^2)$

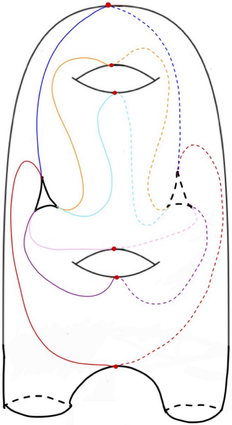
Notation: $(b; m_1, \ldots; k_1, \ldots)$ = (boundary; marked-point; non-marked interior) prongs.

Goal, restated

Classify all pseudo-Anosov maps on these strata lifting to fixed-point-free (FPF) maps on Σ_2^2.

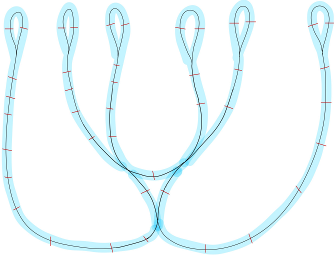
Train tracks on surfaces





A train track is an embedded graph with smooth tangent structure at each vertex (switch).

Train tracks carry pseudo-Anosovs



Thicken the track $\to$ fibered neighborhood, foliated by vertical fibers.

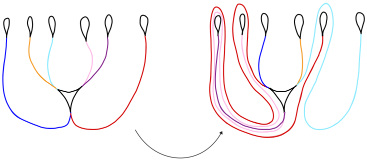
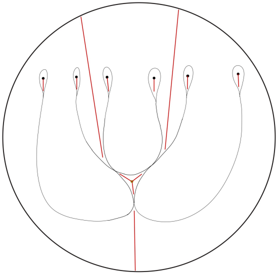


Image of τ runs transverse to the fibers $\to$ induced train track map $f_\tau : \tau \to \tau$.

We say τ carries the pseudo-Anosov.

Singularities and cusps



Singularities sit at the centers of complementary polygons.
k-prong singularity $\Leftrightarrow$ k-sided polygon.
Peripherally: $\#$ cusps along $\partial = \#$ boundary prongs.
Prongs of $\mathcal{F}^u$ emanate from singularity/boundary to cusps.

Why study train track maps?

Theorem (Reinoso 2024; after Bestvina–Handel)

A pseudo-Anosov on D_n carried by τ is determined, up to conjugacy, by its induced train track map $f\colon \tau \to \tau$.

(And the underlying mapping class/braid is determined up to the full twist $\Delta^2 = (\sigma_1 \cdots \sigma_{n-1})^n$)

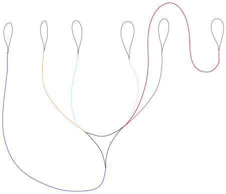
- Same train track map $\Rightarrow$ same pseudo-Anosov.

Classifying pseudo-Anosovs reduces to classifying train track maps.

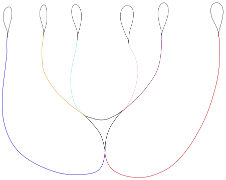
Standard and jointless

Theorem (Farber–Reinoso–Wang)

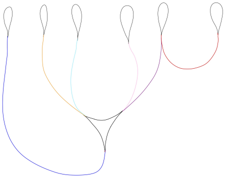
Any pseudo-Anosov on D_n is carried by a standard, jointless train track in the same stratum.



Non-standard



Standard and jointless



Standard with a joint

Standard: Only one edge above each marked point.
Jointless: Every marked-point polygon has no cusp on it.

Enumeration problem

A conundrum: Given a stratum, there are many such train tracks. Can we be sure we have all of them?

$\Rightarrow$ algorithmically enumerate all standard, jointless tracks in a stratum

Orientable lifts of the foliation

Three of the five strata — $(2; 1^4, 2^2; \varnothing)$, $(2; 1^6; 4)$, $(4; 1^6; \varnothing)$ — can be eliminated by an orientability argument.

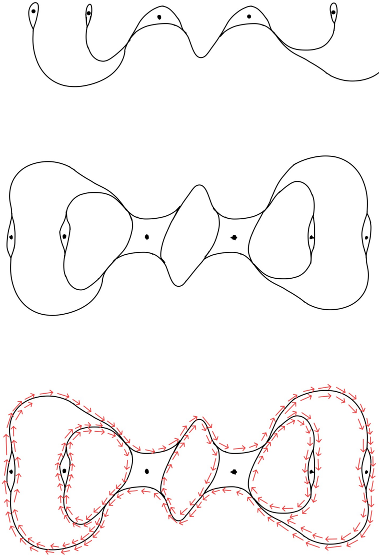
Orientability obstruction (Thurston)

If the lifted foliations on Σ_2^2 are orientable, then Φ acts on $H_1(\Sigma_2^2)$ with the dilatation λ as a real eigenvalue $\Rightarrow \lambda > 1$ is a real root of $\Delta_K(t)$.

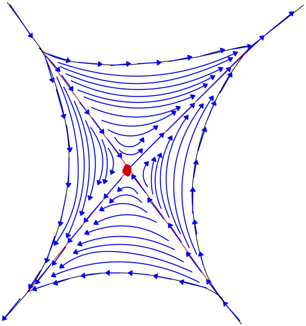
Orientable train tracks

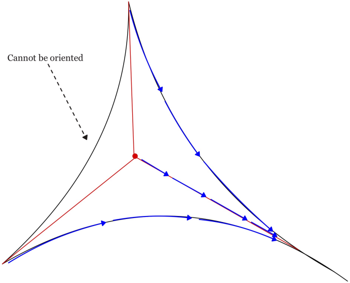
The lifted foliation is orientable in these strata because the train tracks in these strata are orientable.

Orientable train track in $(2; 1^4, 2^2; \varnothing)$



Necessary: Even-sided polygons





4-gon: orientation closes consistently. ✓

3-gon: parity fails. ×

Pick a direction for one leaf; orientations propagate around the polygon. Even sides closes; odd sides forces an inconsistency.

Three down, two to go

$(t^2 - t + 1)^2$ has no real roots — its roots are primitive 6th roots of unity.

$$\text{No real } \lambda > 1. \text{ Contradiction.}$$

The three orientable strata are eliminated.

Eliminated: $(2; 1^4, 2^2; \varnothing), (2; 1^6; 4), (4; 1^6; \varnothing)$
Remaining: $(3; 1^6; 3)$ and $(2; 1^6; 3^2)$. Both are non-orientable.

Goal, restated

Classify all train track maps on standard jointless train tracks
on the two non-orientable strata
lifting to fixed-point-free (FPF) maps on Σ_2^2.

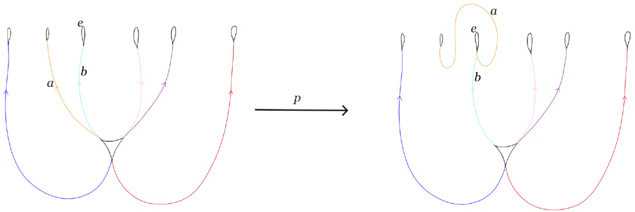
Enumerating train tracks

- We need to enumerate all standard jointless tracks on D_6 in the strata $(3; 1^6; 3)$ and $(2; 1^6; 3^2)$.
- There might be many. Have we gotten them all?
- Idea: Use folding automaton to algorithmically enumerate all standard train tracks, then filter by jointless.

Elementary folding

Theorem (Bestvina–Handel 1995)

Every train track map decomposes as a finite sequence of elementary folds.



At a cusp (a, b) at a switch, fold a onto b. In real-edge words: $p(a) = b \cdot a'$.

The folding automaton

Original idea (Ko–Los–Song 2002; Los 2008)

From a starting standard track τ, perform all possible folds; check if each result is new; repeat. This process terminates and produces an *automaton*, a directed graph.

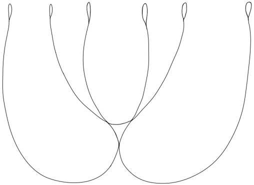
`autofolder` (Y. 2026)

An explicit algorithmic realization of the Ko–Los–Song folding automaton.

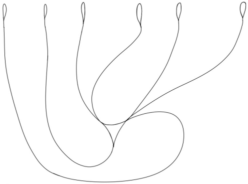
Python on SageMath. Input: a stratum. Output: the full folding automaton.

Stratum	**Standard**	**Jointless**
$(3; 1^6; 3)$	138	4
$(2; 1^6; 3^2)$	110	20

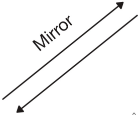
autofolder output, standard jointless tracks on $(3; 1^6; 3)$

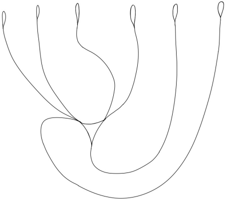


(0.) The candelabra

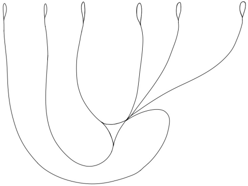


(1.)



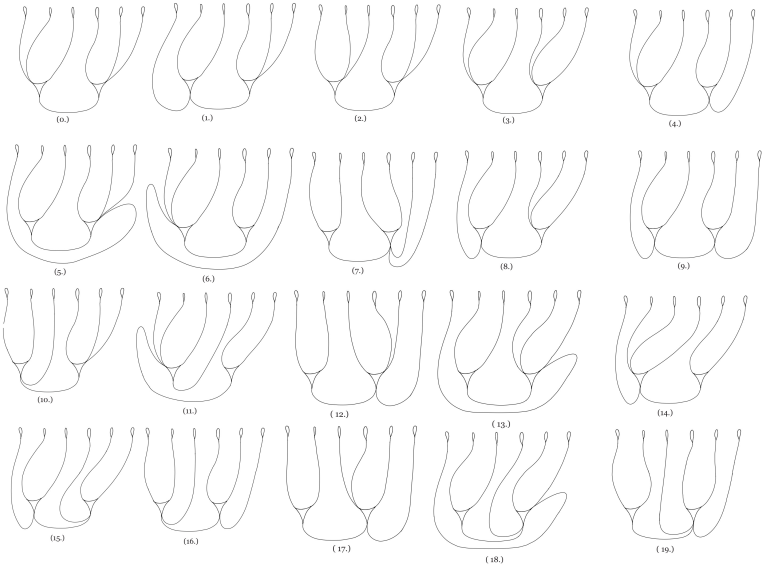


(2.)



(3.)

`autofolder` output, standard jointless tracks on $(2^2; 1^6; 3^2)$

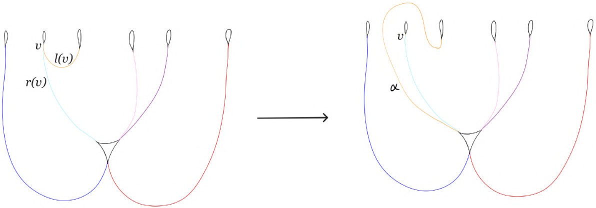


`autofolder` output: standard, jointless, modulo mirrors

Stratum	Standard	Jointless	Mirror
$(3; 1^6; 3)$	138	4	3
$(2; 1^6; 3^2)$	110	20	12

Tight splitting

A split at a switch v is the inverse of a fold.

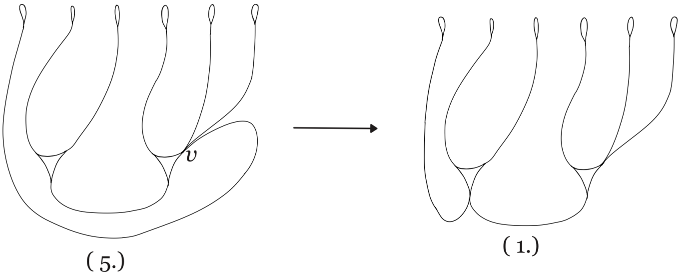


Farber-Reinoso-Wang give a sufficient condition under which after splitting, the resulting track still carries a map the original one carried:

Theorem (Farber–Reinoso–Wang)

A switch of maximum real valence > 1 at a polygon can always be split so that the resulting track still carries the same pseudo-Anosov map (up to conjugacy) the original track does.

Splitting reduces the candidate list



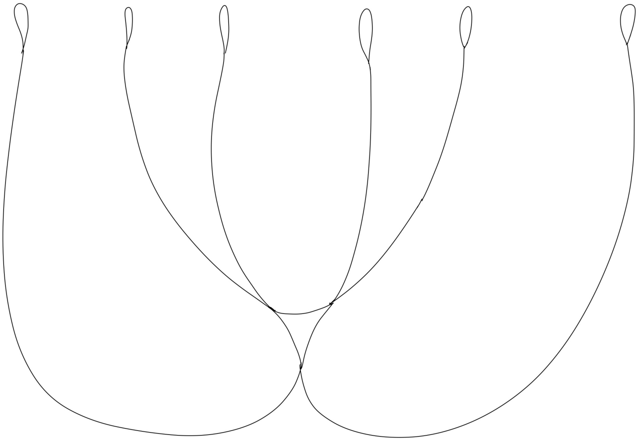
Every pseudo-Anosov carried by 5 is carried by 1.

Reduction: Jointless+Mirror+Splitting

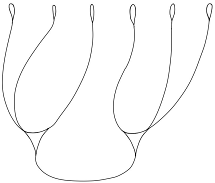
Stratum	Standard	Jointless	Mirror	Splitting
$(3; 1^6; 3)$	138	4	3	1
$(2; 1^6; 3^2)$	110	20	12	6

We are now ready to begin train track map analysis on these tracks.

$(3; 1^6; 3)$, the Candelabra



$(2; 1^6; 3^2)$, six train tracks



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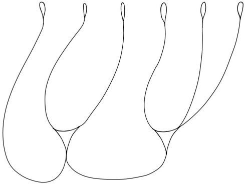
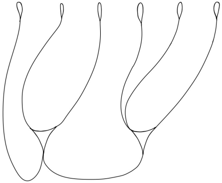
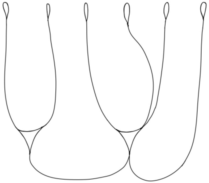


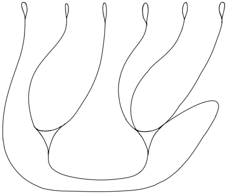
(1.)



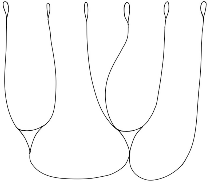
(8.)



(12.)



(13.)



(17.)

Goal, restated once more

Classify all train track maps on the seven tracks above
such that for any pseudo-Anosov inducing these maps, lift to FPF maps on Σ_2^2.

Fixed-point-free: The Trace Lemma

Train track maps on D_6 lift via Birman–Hilden to Σ_2^2. The Trace Lemma constrains which lifts can be FPF upstairs.

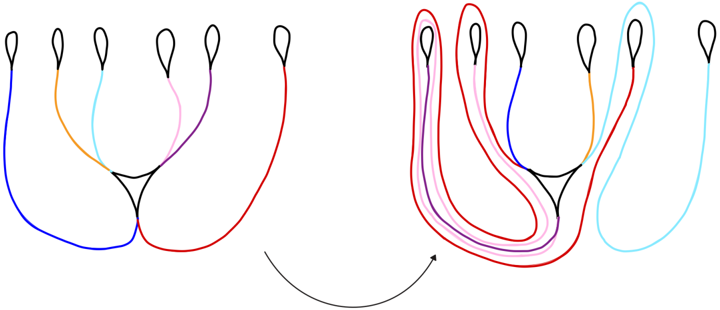
Trace Lemma (original; via Lefschetz)

If the lifted pseudo-Anosov Φ on Σ_2^2 is FPF, then in its induced train track map no edge traces itself (transition matrix is traceless).

Trace Lemma, jointless version downstairs

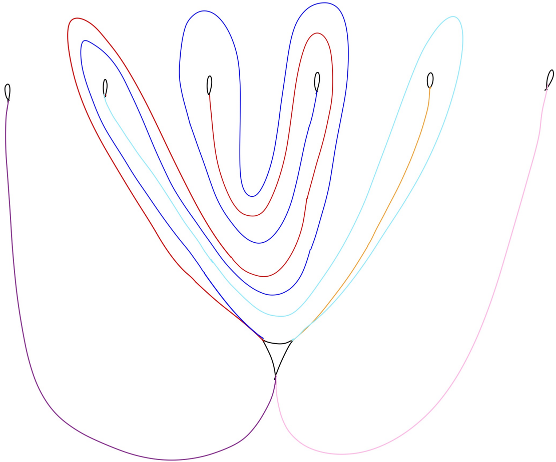
Downstairs: if a real edge e is incident to a jointless monogon, then e does not appear in $f_\beta(e)$.

Trace Lemma in Practice

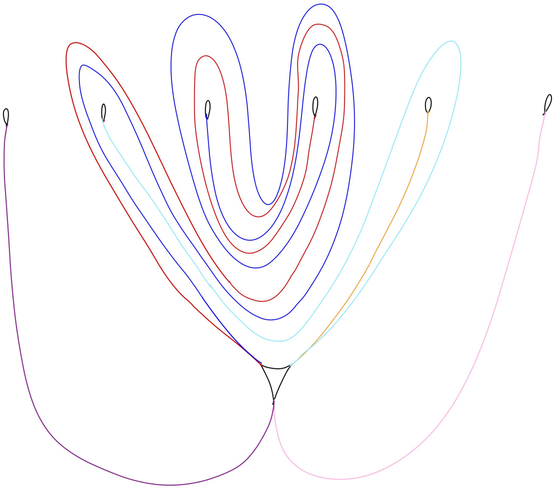


The image of every real edge does not trace itself.

Case analysis example



Case analysis example



Candidate families after case analysis

Stratum	Track	Families
$(3; 1^6; 3)$	Candelabra	38
$(2; 1^6; 3^2)$	Track 0	13
	Track 1	17
	Track 8	9
	Track 12	0
	Track 13	4
	Track 17	6
	subtotal	49
Total		87

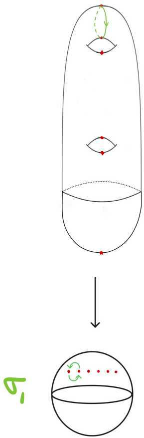
Each family can be associated with a braid family inducing said train track maps.

Goal, restated — in terms of K

For each of the 87 candidate braid families β,
determine whether β can be induced by (via Birman–Hilden) the monodromy of a knot $K \subset S^3$
with $\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$.

Two algebraic filters: $K \subset S^3$ (*F1*) and $\Delta_K = (t^2 - t + 1)^2$ (*F2*).

Homological filtering



- Each braid generator is a Dehn twist upstairs, which on the generators of $H_1(\Sigma_2; \mathbb{Z})$, representable as a matrix. Composing:

$$(\hat{h})_* \; : \; H_1(\Sigma_2; \mathbb{Z}) \to H_1(\Sigma_2; \mathbb{Z}),$$

  a 4×4 integer matrix.

Two algebraic tests follow.

Filter F1: zero-surgery condition

For K to be a knot in S^3, the 0-surgery $S^3_0(K)$ must have

$$H_1(S^3_0(K);\mathbb{Z}) \;\cong\; \mathbb{Z}.$$

Computing H_1 from $\hat{h}_*$ on $H_1(\Sigma^0_2)$, this becomes:

$$\boxed{\left|\det(I-(\hat{h})_*)\right| \;=\; 1}$$

Most candidate families fail F1 immediately — determinant comes out ≥ 2.

Filter F2: Alexander polynomial

The characteristic polynomial of $(\hat{h})_*$ must match the granny's Alexander polynomial:

$$\boxed{\det(tI - (\hat{h})_*) \doteq (t^2 - t + 1)^2}$$

- This eliminates everything F1 didn't.
- Some families pass F1 but fail F2 (or vice versa) at every integer parameter value.

Verdict

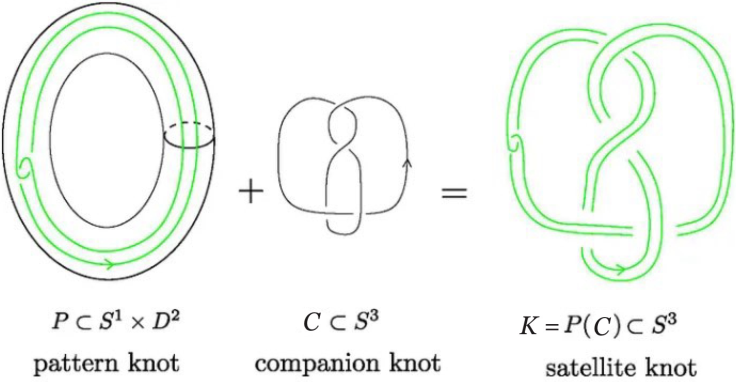
Of the 87 candidate families,

zero pass both F1 and F2.

The hyperbolic case is fully eliminated.

Satellite knot case

Satellite knots, briefly



Schubert's formula relates Δ_K to the Alexander polynomials of P, C.

Schubert's formula

Theorem (Schubert, 1953)

If $K = P(C)$ is a satellite with pattern P, companion C, and winding number w, then

$$\Delta_K(t) \doteq \Delta_{P(U)}(t) \cdot \Delta_C(t^w).$$

Plug in $\Delta_K(t) = (t^2 - t + 1)^2$ and impose fiberedness:

$$P(U) \text{ is a trefoil}, \quad C \text{ is a trefoil}, \quad w = 1.$$

(Symmetric factorization; the alternative $\Delta_{P(U)} = 1$ forces $K = C$ trivially.)

Thurston reducing system

- Monodromy $h \simeq \phi$ reducible.
- Thurston reducing system $\Gamma = \gamma_1, \ldots, \gamma_n$ is set of multi-curves preserved by the map.
- Map fixes outer piece $\phi_0 \colon \Sigma_0 \to \Sigma_0$, and is either periodic or pseudo-Anosov.

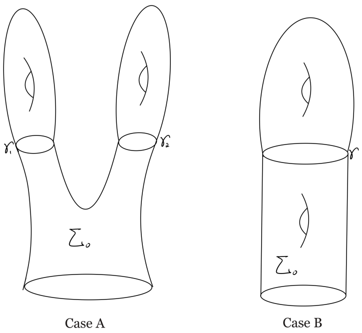


Figure 8.1: the two non-trivial cases of Γ.

Case A: Σ_0 pair of pants

- Pair of pants has no pseudo-Anosov homeomorphisms.
  $\Rightarrow$ outer map h_0 is periodic.
- Baldwin–Sivek (Proposition 2.3 of [BS25]): for periodic outer maps with $\Gamma \neq \emptyset$, K is one of:
  - a torus knot (killed),
  - a (p, q)-cable with Σ_0 of genus $g(T_{p,q}) = 0$ — forces $P(U) =$ unknot, contradiction,
  - composite, with Σ_0 planar.
- Only composite genus-2 fibered knot with $\Delta = (t^2 - t + 1)^2$:

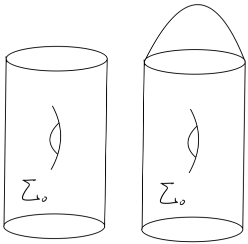
connect-sum of two trefoils.

Case B: Σ_0 is genus-1

Outer map ϕ_0 is periodic or pseudo-Anosov.

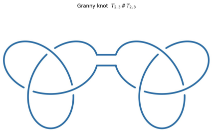
Periodic $\Rightarrow$ (p, q)-cable. Pattern is the trefoil. But $T_{2,3}$ as a cable has winding number 2 or 3, contradicting $w = 1$.

Pseudo-Anosov $\Rightarrow$ capping off γ recovers the trefoil monodromy — not pA. Forces $|c(h)| = 1$ exactly. A contact-invariant argument rules this out.

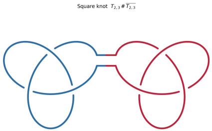


Case B is empty.

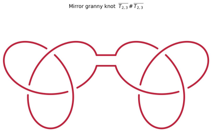
Three candidates remain



Granny $T_{2,3} \# T_{2,3}$



Square $T_{2,3} \# \overline{T}_{2,3}$



Mirror granny $\overline{T}_{2,3} \# \overline{T}_{2,3}$

The trichotomy has narrowed K to a connect-sum of two trefoils.
Three options remain.

The bigrading separates them

The three knots share the same Alexander polynomial and the same total $\widehat{HFK}$-rank, but their bigraded $\widehat{HFK}$'s are pairwise distinct.

Only the granny matches $\widehat{HFK}(K)$.

Main Theorem, proved

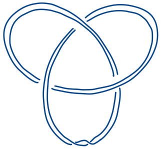
Among the three candidates, only the granny matches
$\widehat{HFK}(K) \cong \widehat{HFK}(T_{2,3} \# T_{2,3})$ as bigraded vector spaces.

Main Theorem (Y.): $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$. ■

(Equivalently, detects $\overline{T_{2,3}} \# \overline{T_{2,3}}$)

Future Direction

A second target



$C_{2,1}(T_{2,3})$: the $(2,1)$-cable of the right-handed trefoil. Genus 2, fibered. Same rank in next-to-top Alexander grading. Alexander polynomial:

$$\Delta_{C_{2,1}(T_{2,3})}(t) \;=\; t^4 - t^2 + 1 \;=\; \Phi_{12}(t).$$

- Different from $(t^2 - t + 1)^2$, so not a Main Theorem candidate.
- Same machinery applies.

Theorem: partial detection of the cable

Theorem (Y.)

If $\widehat{HFK}(K) \cong \widehat{HFK}(C_{2,1}(T_{2,3}))$ as bigraded $\mathbb{F}_2$-vector spaces, then either

- $K \cong C_{2,1}(T_{2,3})$, or
- K is hyperbolic, with disk braid in one of 6 braid families.

Partial detection: residual list not closed by homological filters alone.

Corollary (Y.)

$\widehat{HFK}$ detects $C_{2,1}(T_{2,3})$ among *non-hyperbolic* knots.

Closing the cable gap

Strategies for the 6 residual families:
(A) Check Pseudo-Anosov-ness. Write down and check transition matrix.
(B) Check FPF. Cotton-Clay formula.
(C) Direct knot reconstruction via SnapPy.

This would give first $\widehat{HFK}$ detection of a prime satellite.

Open problem, cable detection

Open problem (Reinoso, 2024)

Prove that $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$ (the granny knot), or find a counterexample to show it does not.

If detection is shown, will be the first detection result of $\widehat{HFK}$ for a prime satellite knot!

Open problem, cable detection

~~Open problem (Reinoso, 2024)~~

~~Prove that $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$ (the granny knot), or find a counterexample to show it does not.~~

✓

If detection is shown, will be the first detection result of $\widehat{HFK}$ for a prime satellite knot!

Open problem, cable detection

~~Open problem (Reinoso, 2024)~~

~~Prove that $\widehat{HFK}$ detects $T_{2,3} \# T_{2,3}$ (the granny knot), or find a counterexample to show it does not.~~

✓

Open problem (Yeh, 2026)

Prove that $\widehat{HFK}$ detects the $(2, 1)$-cable of the trefoil, or find a counterexample to show it does not.

If detection is shown, will be the first detection result of $\widehat{HFK}$ for a prime satellite knot!

Thank you

Knot Floer homology detects the granny knot.

- Advisor: John A. Baldwin.
- Committee, collaborators, family.

How the story ended.

Backup slides

More future directions

Genus ≥ 3: the hyperelliptic constraint

In genus $g \geq 3$, the hyperelliptic mapping class group

$$H_g \;=\; C_{\mathrm{Mod}(\Sigma_g)}(\iota)$$

is a proper subgroup. The framework extends only to *hyperelliptic* monodromies.

If it is known a fibered knot has hyperelliptic monodromy, the strategy can be used.

Beyond two fixed points

What if the monodromy has more than one fixed point? Extend the fact that ι swaps the fixed point and the additional one from capping off:

Lemma (Y.)

For a fibered knot in an integer homology sphere with hyperelliptic-symmetric capped monodromy, ι pairs the fixed points of $\hat{h}$.

Concrete genus-2 hyperbolic fibered targets in S^3 with rank 3: 10_{132}, 10_{145}, $K11n38$.

Train track analysis might be very complicated. Automate?

Khovanov homology detects: unknot, trefoils, figure-eight, $T_{2,5}$, $T_{-2,5}$, 5_2, certain Whitehead doubles — all prime. **Proposed strategy** for the granny (three steps):

1. Bigraded Kh-iso $\Rightarrow$ K is Khovanov-thin.
2. Dowlin's spectral sequence $\Rightarrow$ $\widehat{HFK}(K)$ thin $\Rightarrow$ $\widehat{HFK}(K;\mathbb{Q}) \cong \widehat{HFK}(T_{2,3} \# T_{2,3};\mathbb{Q})$.
3. Extend Main Theorem from $\mathbb{F}_2$ to $\mathbb{Q}$ coefficients.

Would be the first Khovanov detection of a composite knot.

Why the square knot is genuinely harder

Hedden–Watson + Wang: the pretzel family $\{P(3,-3,2n)\}_{n\in\mathbb{Z}}$ contains the square knot at $n=0$, with identical $\widehat{HFK}$ and instanton $\widehat{HFK}$ for every member. $\Rightarrow$ $\widehat{HFK}$ cannot detect the square knot.

A conjectural complementarity

Question. Does $\widehat{HFK}$ detect membership in the pretzel family $\{P(3,-3,2n)\}$? If so:

- $\widehat{HFK}$ would identify the family (modulo HFK-equivalence),
- Khovanov would resolve the specific member.

First known invariant pair coarsely detecting a non-trivial infinite family.

In 2008, Los called for an explicit algorithmic realization of the folding automaton, noting that without one, the state space remains intractable.

`autofolder` answers that call.

Standalone applications of the algorithm:

- Dilatation enumeration via Perron–Frobenius bounds on transition matrices.
- Pseudo-Anosov classification within strata.
- Edge data (standardizing braids, transition matrices) is computed but not yet exposed — activating it enables finer dynamical invariants.
- Closed-surface generalization: would enable detection of fibered knots in 3-manifolds beyond S^3.

Available as a Python / SageMath package.

Six future directions *(for young grad students)*

1. Partial cable detection.
2. Genus ≥ 3 targets (the hyperelliptic constraint).
3. Targets with multiple interior fixed points (next-to-top rank ≥ 3).
4. Khovanov detection of the granny.
5. The Hedden–Watson pretzel family and HFK/Khovanov complementarity.
6. `autofolder`: Enumerate pseudo-Anosovs with bounded dilatation.

A. Cited results

Fixed-point bound: Ghiggini–Spano & Ni — how

Claim

$\dim \widehat{HFK}(K, g-1) = r \;\Rightarrow\;$ the pseudo-Anosov monodromy has $\leq r-1$ interior fixed points.

Ni's route (generalizes Baldwin–Hu–Sivek):

- Connect-sum 0-surgery $S^3_0(K\#K)$: forces the degree-1 class monotone (needs fiber genus ≥ 3; the knot alone fails).
- OS 0-surgery formula computes HF^+ from the $\widehat{HFK}$ ranks.
- Lee–Taubes $PFH \cong \widehat{HM}$; Kutluhan–Lee–Taubes $\widehat{HM} \cong HF$; second LT $PFH \cong HF^{\mathrm{symp}}$.
- Cotton-Clay: HF^{symp} generated by fixed points $\Rightarrow$ rank bounds $|\mathrm{Fix}|$.

Ghiggini–Spano: same bound, intrinsic (stays on the surface).

$\widehat{HFK}$ detects genus and fiberedness — how

Easy directions are formal; the content is the converse, resting on Gabai.

Genus (Ozsváth–Szabó)

$g(K) = \max\{A : \widehat{HFK}(K, A) \neq 0\}$. Vanishing above g: adjunction. Nonvanishing at g: Gabai taut foliation + OS nonvanishing (Eliashberg–Thurston + contact invariant).

Fiberedness (Ni; Ghiggini g=1; Juhász via *SFH*)

Fibered $\iff$ $\dim \widehat{HFK}(K, g) = 1$. Here $\widehat{HFK}(K, g) = SFH$ of the complement cut along the fiber; rank 1 $\iff$ product sutured manifold $\iff$ fibered.

Trefoil: fibered of genus one — without the deep results

Genus $= 1$**.** Seifert's algorithm on the 3-crossing diagram $\to$ once-punctured torus $\Rightarrow g \leq 1$; not unknot $\Rightarrow g \geq 1$. **Fibered** (any one suffices):

- Stallings: closure of the positive braid σ_1^3;
- open book $(\Sigma_1^1, T_a T_b)$; monodromy periodic of order 6 $\Rightarrow$ torus knot.

Two corollaries

- First $\widehat{HFK}$-detection of a composite knot.
- Combined with Binns: $\widehat{HFK}$ detects the unique composite almost L-space knot (Corollary 1.8).

B. Knots and braids

Homogeneous braids (and Stallings)

Definition

A braid word is homogeneous if each generator σ_i appears with only one sign throughout. (Positive braids $\subset$ homogeneous; different generators may differ in sign.)

Examples. σ_1^3 ✓ $\sigma_1^2\sigma_2^{-3}\sigma_1\sigma_2^{-1}$ ✓ $\sigma_1\sigma_2\sigma_1^{-1}$ (not: σ_1 has both signs).

Stallings (1978)

The closure of a homogeneous braid is fibered, with fiber the Seifert-algorithm surface.

Property of the *word*, not the link. Trefoil $= \widehat{\sigma_1^3}$ is homogeneous $\Rightarrow$ fibered, fiber a once-punctured torus.

Mirror equivalence of detection

Mirror flips bigrading signs

$\widehat{HFK}_m(\overline{K}, a) \cong \widehat{HFK}_{-m}(K, -a)$ — mirroring = grading reversal R.

Detection equivalence. If $\widehat{HFK}$ detects $T_{2,3}$ and $\widehat{HFK}(J') \cong \widehat{HFK}(\overline{T}_{2,3})$:

$$\widehat{HFK}(\overline{J'}) \cong R\,\widehat{HFK}(J') \cong \widehat{HFK}(T_{2,3}) \;\Rightarrow\; \overline{J'} \cong T_{2,3} \;\Rightarrow\; J' \cong \overline{T}_{2,3}.$$

But *not the same data*: opposite $\tau = \pm 1$, opposite δ-diagonal.

Gives the mirror granny for free; the square knot $T_{2,3} \# \overline{T}_{2,3}$ (opposite chiralities) does not apply.

C. Topology to dynamics

Why ι swaps z and p — detailed

Suppose ι fixed both z and p. Then both become Weierstrass points; descending, h would lift a 5-braid β on D_5. Since $\hat{\phi}_h$ fixes z and z descends to a branch point, the braid permutation σ_β has a fixed index $\Rightarrow$ closure $\hat{\beta}$ has $\mu(\hat{\beta}) \geq 2$ link components. The mapping torus M_h is the branched double cover of S^3 along $\hat{\beta}$:

$$M_h \;\cong\; \Sigma\big(S^3, \hat{\beta}\big).$$

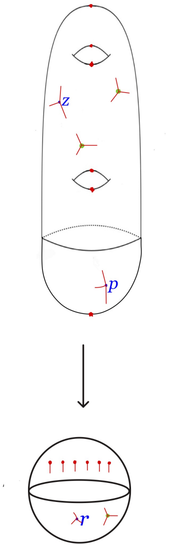
For a μ-component link, $b_1\big(\Sigma(S^3, \hat{\beta})\big) \geq \mu - 1 \geq 1$. But $K \subset S^3$ a knot $\Rightarrow$ $M_h \cong S^3 \Rightarrow b_1(M_h) = 0$.

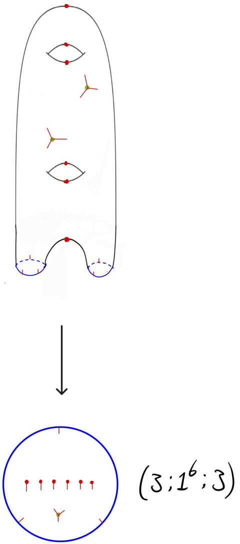
Contradiction. So ι swaps z and p.

Capping off: the new fixed point and the prong count

Why a new fixed point appears. h fixes $\partial\Sigma_2^1$ pointwise $\Rightarrow$ the capping disk is fixed setwise; the k boundary prongs converge to the center p, a k-pronged singularity rotated by $\hat{h}$ $\Rightarrow$ p is a new fixed point. (h has z; $\hat{h}$ has $\{z, p\}$, swapped by ι.)

Prong configuration example $\{3, 3, 3, 3\}$





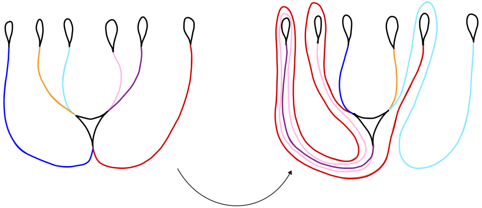
D. Train tracks & the `autofolder`

Train tracks: Thurston, Penner–Harer, Bestvina–Handel

- Thurston — invented them to *carry* measured laminations; coordinatize the Thurston boundary; dilatation = Perron–Frobenius eigenvalue of the carrying matrix.
- Penner–Harer — the rigorous *combinatorics*: weights, switch conditions, splitting moves; $\mathcal{PMF}$ is a sphere in these coordinates.
- Bestvina–Handel — made it *algorithmic* via folding: efficient representatives determine Nielsen–Thurston type and dilatation (originating in $\mathrm{Out}(F_n)$).

`autofolder` lives in the third tradition: Bestvina–Handel folding, automated as an exhaustive enumeration over a stratum.

Transition matrix



	blue	orange	sky blue	pink	purple	red
blue	0	0	1	0	0	0
orange	0	0	0	0	1	0
sky blue	0	0	0	0	2	1
pink	2	1	0	0	0	0
purple	1	0	0	0	0	0
red	2	2	1	0	1	0

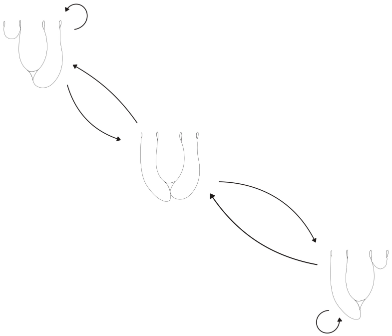
Transition matrices count traversals of real edges only.

The folding automaton: example

Vertices: standard tracks Edges:

elementary folds. Every

pseudo-Anosov in the stratum is a cycle in the automaton (Ko–Los–Song).



Folding automaton of stratum $(1; 1^4; 3)$ on D_4.

Three discrete-combinatorial reformulations turn the algorithm into running code:

- Data structure. Encode a train track as a combinatorial graph decorated with cusp data and cyclic edge orderings at each switch.
- Isomorphism check. Decide whether two tracks are isomorphic by a graph-isomorphism comparison on the decorated data — no continuous deformations to track.
- Folds as rewrites. Elementary folds become local discrete rewrite rules on the data structure; standardizing braids are computed from the cusp updates.

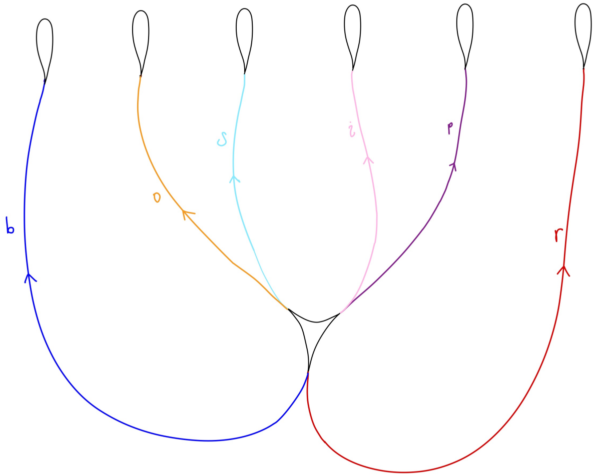
Implemented in Python on SageMath. Input: a stratum. Output: vertices, edges, and braid labels of the full folding automaton.

pA classification

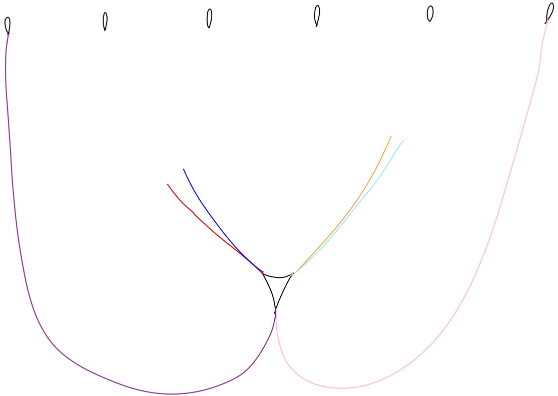
Theorem (Y.)

Every FPF pseudo-Anosov on Σ_2^2 in either non-orientable stratum is conjugate to the lift of one of 87 explicit braid families, up to composition with full twists and mirroring.

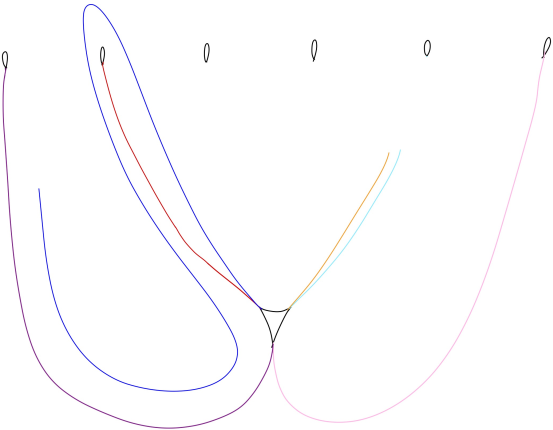
Case analysis example



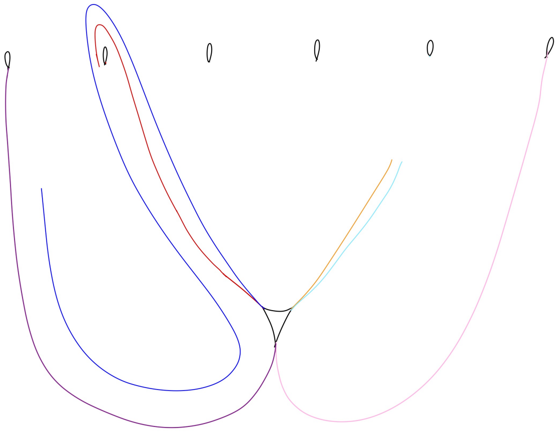
Case analysis example



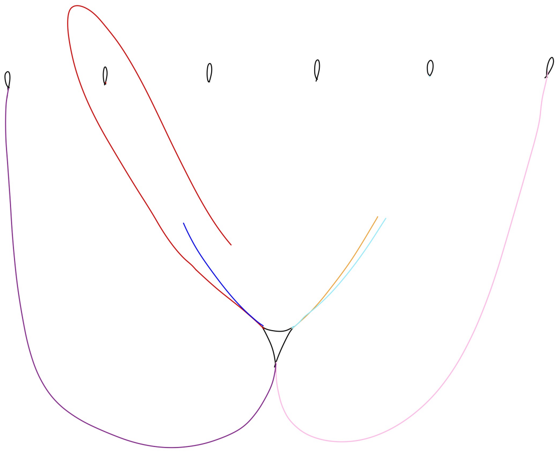
Case analysis example



Case analysis example

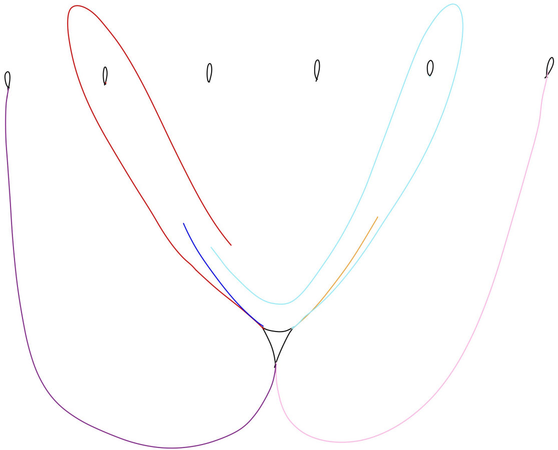


Case analysis example



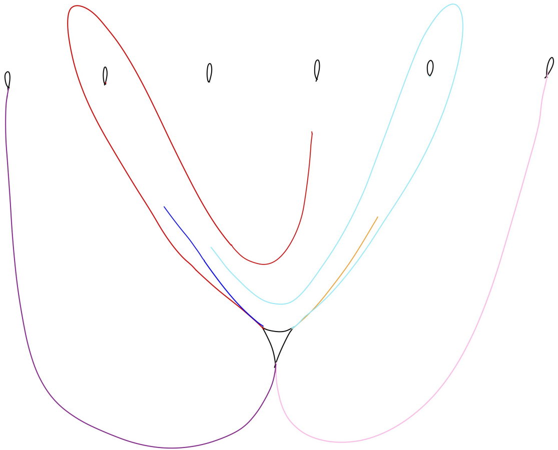
Case analysis example

Case analysis example



Case analysis example

Case analysis example



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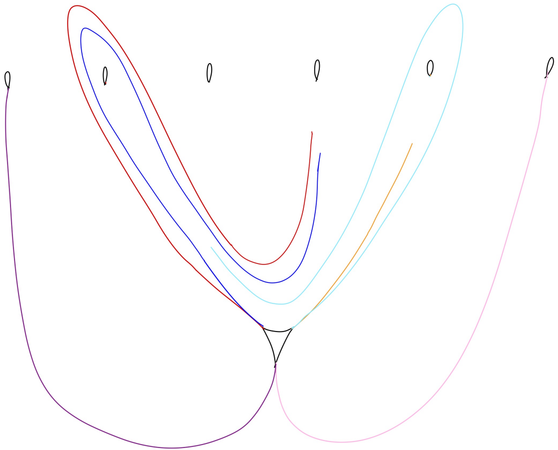

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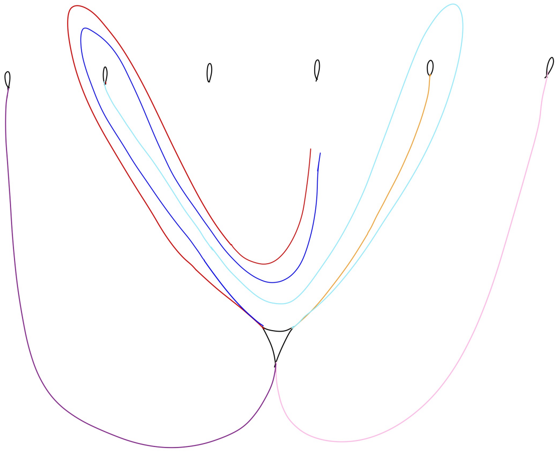
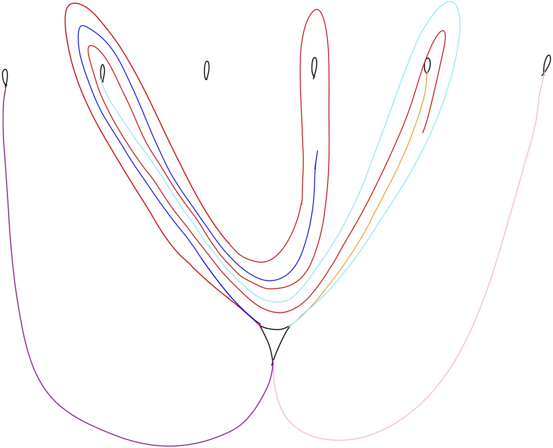
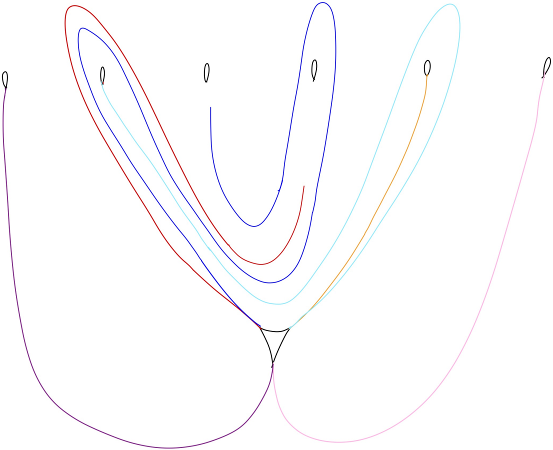


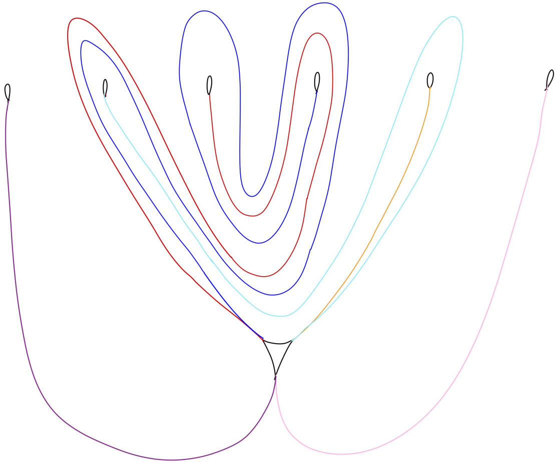
Case analysis example



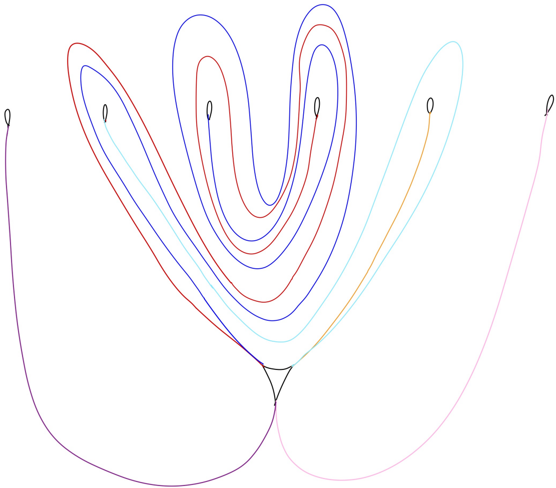
Case analysis example



Case analysis example



Case analysis example



E. Splitting & the Los correspondence

Los: the loop *is* the mapping class

Two resolutions of one monodromy.

- Train-track map $f_\tau : \tau \to \tau$ = the closed form (matrix, dilatation λ, invariant measure μ).
- A loop = the same monodromy *factored* into elementary splits + standardizing braids; one lap = one application of β (renormalizes μ by λ^{-1}).

Eventually periodic. A single pA threads a deterministic path (its measure picks the unique tight split at each switch); a deterministic walk on a finite graph = a finite pre-period (aligning the start to the canonical track) + a cycle (the pA), the pre-period absorbed into a conjugation.

Continued-fraction analogy: purely periodic (start on loop) vs. eventually periodic with a finite prefix.

Splitting termination (1): why weight-descent fails

Goal. Every tight-splitting sequence reaches a terminal track. The splitting diagram is a *finite* directed graph (nodes = iso classes, by `autofolder`). The only obstruction to termination is a directed cycle — a splitting loop.

The correction

A strictly decreasing positive weight sequence need not reach 0; and the renormalization sequence of a pseudo-Anosov *is* an infinite periodic loop, scaling the measure by λ^{-1} each lap.

Weights $\to 0$ is the *defining feature* of a pseudo-Anosov, not a contradiction.
Termination must be argued loop-by-loop.

Splitting termination (2): two loop types

Los. The loop braid γ_C = monodromy, up to conjugacy, full twist, power ⇒ a pA traversing C forever has $\phi_\beta \cong \gamma_C$ (same NT type; λ a power).

Type 1 — jointless loop

$\gamma_C = \sigma_2^{-1}\sigma_1^{-1} = \delta^{-1}$ is periodic ⇒ not pA ⇒ $\lambda = 1$, contradicting $\lambda > 1$. Sequence exits.

Type 2 — jointed loop

Integer joint number $J \geq 1$, bounded below; Reinoso's Theorem C + the per-split weight drop ⇒ J strictly decreases ⇒ exits.

Two loop types, both dispatched; no third type occurs ⇒ termination.

F. The $(2, 1)$-cable

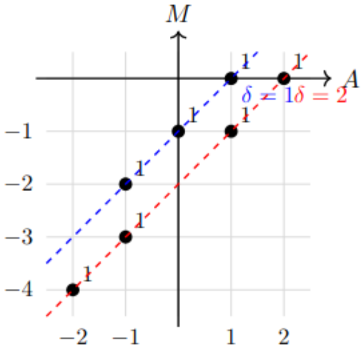
The same pipeline runs

- **Torus.** L-space rank bound vs. $\dim \widehat{HFK}(C_{2,1}(T_{2,3}), 1) = 2$. Killed.
- **Satellite.** Schubert factorization of $\Phi_{12}(t)$ forces $K = C_{2,1}(T_{2,3})$. Done.
- **Hyperbolic.** Same five strata. Orientability still kills the three orientable strata (no real roots of Φ_{12}). Same 87 candidate braid families from case analysis. Apply filters with F2 retargeted:

$$\det(tI - A_{\mathrm{red}}) \;=\; t^4 - t^2 + 1.$$

Result: $87 - 6 = 81$ families killed, $\boxed{6}$ residuals survive.

$\widehat{HFK}$ of the cable



$\widehat{HFK}(C_{2,1}(T_{2,3}); \mathbb{F}_2)$

Cable: the satellite case

Not eliminated — identified. The cable *is* a satellite ($w = 2$), so we show any genus-2 fibered satellite with $\Delta_K = t^2 - 1 + t^{-2}$ is the cable.

- Schubert: $\Delta_K = \Delta_{P(U)} \cdot \Delta_C(t^w)$; degree match $\Rightarrow$ 3 cases. Two die (Hirasawa–Murasugi–Silver trivial satellite; $b^2 = 3$).
- Survivor: $w = 2$, $C = T_{2,3}$, $P(U) =$ unknot $\Rightarrow$ planar outer piece (pair of pants) $\Rightarrow$ ϕ_0 periodic.
- Baldwin–Sivek: torus / composite / cable. Torus & composite excluded ($t^{\pm 1}$-coeffs $-2, 4, -6 \neq 0$) $\Rightarrow$ cable, $(2,1)$ pattern $\Rightarrow$ $K \cong C_{2,1}(T_{2,3})$.

Backstop: $|b| \geq 3 \Rightarrow T_{2,b}$ is L-space $\Rightarrow$ cabling stays L-space (Hedden) $\Rightarrow$ contradicts the rank-2 grading.

Open problem, cable detection

Open problem (Yeh, 2026)

Prove that $\widehat{HFK}$ detects the $(2,1)$-cable of the trefoil, or find a counterexample to show it does not.