Lagrangian Floer Homology

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Contents

Chapter 0

Overview

Lecture 1: 18 Jan

There are three parts to this course:

1. Symplectic Topology and Geometry
2. Hamiltonian Floer Homology
3. Lagrangian Floer Homology

0.1 Symplectic Topology and Geometry

Definition 1. A *symplectic manifold* is a smooth manifold M equipped with a symplectic form $\omega \in \Omega^2(M)$ i.e. a 2-form such that

1. $\mathrm{d}\omega = 0$, and
2. ω^n is a volume form. Equivalently, that ω is *non-degenerate*. Note here that $2n = \dim M$, forcing the dimension of M to be even.

Example. $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, equipped with the *standard symplectic form*

$$\omega_{st} = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Theorem 1 (Darboux)**.** Locally, every symplectic manifold is "modelled by" $(\mathbb{R}^{2n}, \omega_{st})$.

Darboux's Theorem is an instance of so-called *symplectic flexibility*. Here are some more examples of symplectic manifolds:

Example. Surfaces equipped with area forms.

Example. Kaler manifolds, which are certain types of complex manifolds. Examples include $(\mathbb{C}P^n, \omega_{FS})$, projective varieties. For these examples, there exists an almost complex structure, i.e. an endomorphism

$$J\colon TM \to TM$$

of the tangent bundle TM, such that $J^2 = -\operatorname{Id}$. And the *Kalher condition* is satisfied: That

$$g(v, w) := \omega(v, Jw)$$

defines a metric. In this way, the collection (ω, J, g) are related to each other, and we call J a *compatible almost complex structure* to ω.

It is a fact that compatible alsmost complex structures exist in abundance on symplectic manifolds. They provide an important auxiliary piece of data, like having a metric on a smooth manifold allows one to do Morse Theory.

Why study symplectic manifolds? The original motivation is from Hamilton's take on classical mechanics. Symplectic manifolds generalize the notion of phase space.

Hamiltonian dynamics begins with a symplectic manifold (M, ω), and a smooth function $H\colon M \to \mathbb{R}$ called the *Hamiltonian*. There is then an associated vector field X_H. Particles move along flow lines of X_H. We shall study the trajectory of a particle

$$\gamma\colon \mathbb{R} \to M$$

and its velocity vector at some time t:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma(t).$$

The vector field X_H is defined by specifying:

$$\mathrm{d}H = \omega(X_H, \cdot).$$

Compare this definition to how $\nabla_g H$, the gradient vector field, is defined.

Arnold's Conjecture: Suppose (M, ω) is a symplectic manifold, and $H_t\colon M \to \mathbb{R}$ is a time-dependent smooth function (a Hamiltonian), i.e. $H_t\colon M \times \mathbb{R} \to \mathbb{R}$. Question: How many points $p \in M$ will return to their original position after a unit of time under this Hamiltonian flow? (so we are really asking for number of fixed points)

Conjecture: the number of such points equals the number of fixed points of ϕ_H^1, the time-1 flow of $H = H_t$, and it is conjectured that this quantity is greater or equal to the dimension of $H_*(M, \mathbb{R})$.

Compare this with Lefschetz Fixed Point Theorem:

$$\#\operatorname{Fix}(\phi_H^1) \geq |\chi(X)|$$

Here, by definition $\chi(M)$ is some alternating sum, so its size is smaller than the total dimension of $H_*(M, \mathbb{R})$. This shows that a symplectic manifold should give us a lot more fixed points than usual.

Another point of comparison is with the Morse Inequality: A Morse function $f\colon M \to \mathbb{R}$ is a smooth function with some additional properties. We have the *Morse Inequality* (some version of):

$$\#\operatorname{Crit}(f) = \#\{p \in M \colon \mathrm{d}f_p = 0\} \geq \dim H_*(M;\mathbb{R}).$$

This comes freom Morse homology where the chain complex is generated by $\operatorname{Crit}(f)$, and the homology recovers $H_*(M;\mathbb{R})$.

0.2 Hamiltonian Floer Homology

Floer's idea: to do Morse Theory on the loop space $\mathcal{L}M$, the spaced of contractible loops of M (i.e. $f\colon S^1 \to M$), together with a choice of "Morse function", taken to be the symplectic action functional $\mathcal{A}_H$, which depends on the Hamiltonian H. So that we have

$$\mathcal{A}_H \colon \mathcal{L}M \to \mathbb{R}.$$

Then the critical points of $\mathcal{A}_H$ will be in one-to-one correspondence with $Fix(\phi^1_H)$.

Chapter 2

Symplectic Geometry

Lecture 2: 20 Jan

2.1 Hamiltonian Mechanics

We begin with Newton's Second Law of Motion: A force F acts of a particle with mass m satisfies:

$$F = ma.$$

Here we consider the force to be

$$F\colon \mathbb{R}^3 \to \mathbb{R}^3$$

then the acceleration is the second derivative

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

of $x = (x_1, x_2, x_3)$ with respect to time. Now for simplicity we take $m = 1$, so we have

$$F = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}.$$

This is a single second order ODE. Now let us define *momentum* to be

$$y = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Now we have two first order ODEs:

$$y = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad F = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Let us assume that F is a *conservative* force, i.e.

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = \left(-\frac{\partial V}{\partial x_1}, -\frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}]\right)$$

for some

$$V\colon \mathbb{R}^3 \to \mathbb{R}$$

called the *potential energy.*

We can define the total energy, or the *Hamiltonian* as

$$\begin{aligned} H\colon \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R} \\ (x,y) &\mapsto V(x) + \frac{1}{2} y \cdot y \end{aligned}$$

Here we interpret the x as the position, and y the momentum; and the domain of this map is called the *phase space*; the $V(x)$ as the *potential energy*, and $\frac{1}{2} y \cdot y$ as the *kinetic energy*.

Then Newton's Laws are equivalent to

$$\begin{aligned} \frac{\partial H}{\partial x} &= \frac{\partial V}{\partial x} = -F = -\frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\partial H}{\partial y} &= y = \frac{\mathrm{d}x}{\mathrm{d}t}. \end{aligned}$$

We will take the two equations and together they are known as Hamilton's equations:

Definition 2 (Hamilton's Equations)**.**

$$\frac{\partial H}{\partial x} = -\frac{\mathrm{d}y}{\mathrm{d}t}, \quad \frac{\partial H}{\partial y} = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

We may interpret Hamilton's equations as being a restatement of Newton's Second Law for an object subject to a conservative force.

Now suppose we have a particle moving along a trajectory, and the following is its motion in phase space:

$$\gamma(t) = (x(t), y(t)) \in \mathbb{R}^{2n}$$

in the presence of a Hamiltonian H. Then the following expresses how the total energy of the particle changes as it moves:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(H \circ \gamma(t)) &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial H}{\partial y_i}\frac{\mathrm{d}y_i}{\mathrm{d}t} \quad \text{(chain rule)} \qquad (2.1) \\ &= \sum_{i=1}^{n} -\frac{\mathrm{d}y_i}{\mathrm{d}t} \cdot \frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\mathrm{d}x_i}{\mathrm{d}t} \cdot \frac{\mathrm{d}y_i}{\mathrm{d}t} \quad \text{(Hamilton's equations)} \qquad (2.2) \\ &= 0. \qquad (2.3) \end{aligned}$$

Thus H, the total energy, is conserved as the particle moves.

If γ is the trajectory in phase space, then its tangent vector at a given point is

$$\begin{aligned} \frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial}{\partial x_i} + \frac{\mathrm{d}y_i}{\mathrm{d}t} \cdot \frac{\partial}{\partial y_i} \qquad (2.4) \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial y_i} \cdot \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial}{\partial y_i}. \qquad (2.5) \end{aligned}$$

(here $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ are the basis vectors.) So we have a vector field that depends on H, evaluated on $\gamma(t)$. So we can define the vector field X_H by the formula

$$X_H := \sum_{i=1}^{n} \frac{\partial H}{\partial y_i} \cdot \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial}{\partial y_i}.$$

Thus a trajectory in phase space, i.e. an integral curve of X_H, must satisfy the first-order ODE

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma(t).$$

We want to abstract this mechanism

$$\begin{aligned} C^\infty(\mathbb{R}^{2n}) &\to \Gamma(\mathbb{R}^{2n}) \\ H &\mapsto X_H \end{aligned}$$

for describing physics of motion in the presence of H. Compare this mechanism with the differential of H:

$$\mathrm{d}H = \sum \frac{\partial H}{\partial x_i} \cdot \mathrm{d}x_i + \frac{\partial H}{\partial y_i} \cdot \mathrm{d}y_i.$$

This looks like X_H but they differ because $\mathrm{d}H$ is a 1-form.

The "mediator" between $\mathrm{d}H$ and X_H is given by the 2-form

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \in \Omega^2(\mathbb{R})$$

in the sense that

$$\mathrm{d}H = \omega_0(X_H, -) = \iota(X_H)\omega_0.$$

Definition 3. A linear transformation $A \in \mathrm{GL}(\mathbb{R}^{2n})$ is *symplectic* if $A^*\omega_0 = \omega_0$. In other words, $A\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic if

$$A^*\omega_0(v, w) := \omega_0(Av, Aw) = \omega_0(v, w)$$

for all $v, w \in \mathbb{R}^{2n}$.

Definition 4. A diffeomorphism $\psi\colon U \to V$ of open sets $U, V \subset \mathbb{R}^{2n}$ is *symplectic* if

$$\mathrm{d}\psi\colon T_pU \to T_{\psi(p)}V$$

is a symplectic linear transformation for all $p \in U$.

The set of all symplectic transformations on $\mathbb{R}^{2n}$ is denoted by $\mathrm{Sp}(2n)$, and is called the *symplectic linear group.*

On the other hand, we also have

$$\mathrm{Symp}(\mathbb{R}^{2n}) := \{\psi \in \mathrm{Diff}(\mathbb{R}^{2n})\colon \mathrm{d}\psi_p \in \mathrm{Sp}(2n) \quad \forall p \in \mathbb{R}^{2n}\}.$$

And it is a fact that $\mathrm{Symp}(\mathbb{R}^{2n})$ is a subgroup of $\mathrm{Diff}(\mathbb{R}^{2n})$.

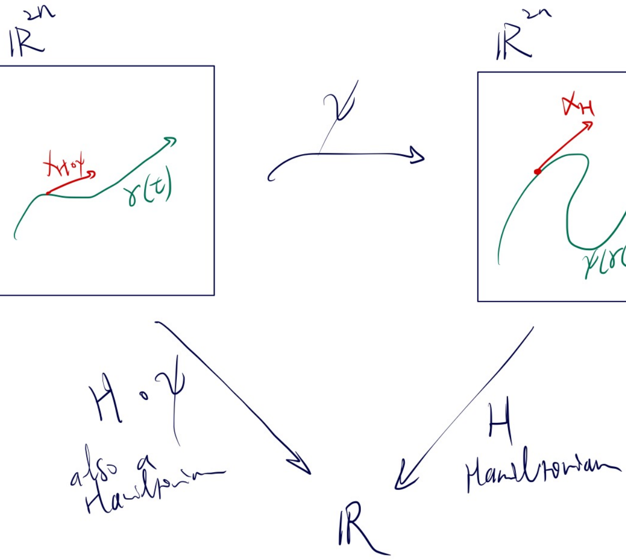

Figure 2.1: Symplectic diffeomorphism of $\mathbb{R}^{2n}$

Intuition. The slogan is: *Symplectic transformations preserve Hamiltonian dynamics.*

More precisely,
That is, $\psi \in \operatorname{Symp}(\mathbb{R}^{2n})$ if and only if for all $H \in C^\infty(\mathbb{R}^{2n})$,

$$\psi^* X_H = X_{H\circ\psi}.$$

Lecture 3: 23 Jan

2.2 Symplectic Manifolds

Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $H \in C^\infty(\mathbb{R}^{2n})$, a "Hamiltonian". We arrived at the associated vector field

$$X_H := \sum_{i=1}^{n} \frac{\partial H}{\partial y_i} \cdot \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial}{\partial y_i},$$

whose dynamics governs the motion of a particle in phase space in the following sense:

Trajectories/integral curves/flow-lines are given by $\gamma\colon \mathbb{R} \to \mathbb{R}^{2n}$ satisfying

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma(t).$$

We would like to generalize to other smooth manifolds. One way is the following: Suppose we have a smooth $2n$-manifold, and $H \in C^\infty(M)$. On a chart

$$\phi\colon M \supset U \to V \subset \mathbb{R}^{2n},$$

consider the function $H \circ \phi^{-1} \in C^\infty(V)$, which is a Hamiltonian in the earlier sense, so we can define $X_{H\circ\phi^{-1}}$ on V as before.

However, this is a local definition. We would like to have a global vector field $X_H \in \Gamma(M)$ such that the restriction on charts is the local definition:

$$X_H\Big|_U = \phi^*(X_{H\circ\phi^{-1}}).$$

In order for this to work, we need to ensure the overlaps are compatible in sense, so we will need a condition on the transition maps between charts.

Theorem 2. This assignment can be made

$$\begin{aligned} C^\infty(M) &\to T(M) \\ H &\mapsto X_H \end{aligned}$$

if and only if the transition functions are symplectic (as diffeomorphisms between open sets of $\mathbb{R}^{2n}$).

Definition 5. A *symplectic manifold* is a smooth manifold M covered by an atlas where the transition maps are symplectic.

Recall that on $\mathbb{R}^{2n}$ there is the 2-form

$$\omega_0 = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

This 2-form is

- Closed, i.e. $\mathrm{d}\omega_0 = 0$; and
- non-degenerate, i.e. ω_0^n is a volume form on $\mathbb{R}^{2n}$.

If M is a symplectic manfiold as defined above, then we get a global 2-form $\omega \in \Omega^2(M)$ by defining locally

$$\omega\Big|_U = \phi^*\omega_0$$

on each chart. We call ω a *symplectic form* on M.

As before, given $H \in C^\infty(M)$, we can define the unique vector field X_H via

$$\mathrm{d}H = \omega(X_H, -)$$

using the non-degeneracy of ω.

For small time t in any neighborhood, we can integrate X_H to get a diffeomorphism $\phi_H^t\colon M \to M$, which is the time-t flow of the vector field X_H.

Proposition 1. Some key facts about ϕ_H^t:

1. It preserves H (conservation of energy).

2. It preserves ω, i.e. $(\phi_H^t)^*\omega = \omega$ (Hamiltonian flow preserves symplectic form).

Proof. 1. For any flow line $\gamma\colon \mathbb{R} \to M$ of X_H, the value of $H \circ \gamma(t)$ is constant:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H \circ \gamma(t) &= \mathrm{d}H\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right) && (2.6)\\ &= \mathrm{d}H(X_H \circ \gamma(t)) && (2.7)\\ &= \omega(X_H, X_H \circ \gamma(H)) && (2.8)\\ &= 0. && (2.9) \end{aligned}$$

2. It suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_H^t\right)^*\omega = 0$$

since

$$(\phi_H^0)^*\omega = \mathrm{Id}^*\,\omega = \omega.$$

By differential geometry,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_H^t\right)^*\omega = \mathcal{L}_{X_H}\omega$$

then by Cartan's magic formula, the RHS is

$$\begin{aligned} \mathcal{L}_{X_H}\omega &= \iota(X_H)\mathrm{d}\omega + \mathrm{d}(\iota(X_H)\omega) && (2.10)\\ &= \mathrm{d}(\iota(X_H)\omega) && (2.11)\\ &= \mathrm{d}\mathrm{d}H && (2.12)\\ &= 0. && (2.13) \end{aligned}$$

(See da Silva)

□

Definition 6. A *symplectic manifold* is a smooth manifold M equipped with a symplectic form.

Theorem 3 (Darboux)**.** If ω is a symplectic form on M and $p \in M$, then there exists a neighborhood U of p in M and a diffeomorphism

$$\phi\colon \mathbb{R}^{2n} \supset V \to U$$

such that

$$\phi^*\omega = \omega_0.$$

2.3 Darboux's Theorem and Moser's Method

Lecture 4: 25 Jan

Theorem 4 (Linear Darboux's Theorem)**.** Let V be a finite-dimensional vector space, and $\Omega \in \Lambda^2 V^*$. The following are equivalent:

1. $\dim V = 2n$ and $\Omega^n \in \Lambda^{2n} V^*$ is non-zero.
2. If $v \in V$ and $\Omega(v, w) = 0$ for all $\omega \in V$, then $v = 0$.
3. The map

$$\begin{aligned} V &\to V^* \\ v &\mapsto \omega(v, -) \end{aligned}$$

   is an isomorphism.
4. There exists a basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ of V such that

$$\Omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

We say Ω is *non-degenerate* if it satisfies any one of the above.

Proof. Proof sketch of 2. ⇒ 4. Suppose $k > 0$ and there exists. □

Proof of Darboux's Theorem

Proof. To begin, choose any neighborhood U of p and a diffeomorphism $U \to U' \subset \mathbb{R}^{2n}$. Now push forward ω to U'. For the remainder, we will denote U' as U, which we require to contain 0, and we will denote the pushforward of ω as simply ω. So now the setting is $U \subset \mathbb{R}^{2n}$, our point of interest is the origin, and ω is a form on this neighborhood of $\mathbb{R}^n$. In particular ω gives a pairing on the tangent space above each point. We put our attention on $\omega\big|_{T_0 U}$, which is non-degenerate in the sense of linear Darboux above.

By linear Darboux, there exists a diffeomorphism

$$\begin{aligned} \phi \colon U &\to U' \subset \mathbb{R}^{2n} \\ 0 &\mapsto 0 \end{aligned}$$

such that

$$\phi_* \omega\Big|_{T_0 U} = \omega_0\Big|_{T_0 U'}.$$

Key idea: Find $\phi = \phi_1$, where ϕ_t is the time-t flow of a time-dependent vector field X_t on U for $0 \le t \le 1$. Moreover, if we define

$$\omega_t = t\omega + (1-t)\omega_0, \quad 0 \le t \le 1$$

we will arrange such that

$$\phi_t^*\omega_t = \omega_0. \tag{2.14}$$

In other words, we claim that there exists a time-dependent vector field X_t, whose isotopy ϕ_t it generates, satisfies 2.14.

Recall (da Silva pg.35) that X_t generating ϕ_t means the following are satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(p) = X_t \circ \phi_t(p).$$

And we would like this isotopy to satisfy $\phi_0 = \mathrm{Id}$.

In order for 2.14 to be satisfied, it amounts to show the isotopy ϕ_t satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\omega_t = 0, \quad 0 \le t \le 1.$$

On the other hand, suppose this isotopy ϕ_t is generated by X_t, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\omega_t \overset{\text{chain rule}}{=} \frac{\mathrm{d}}{\mathrm{d}s}\phi_s^*\omega_t\bigg|_{s=t} + \frac{\mathrm{d}}{\mathrm{d}s}\phi_s^*\omega_s\bigg|_{s=t} \tag{2.15}$$

$$= \phi_t^*\mathcal{L}_{X_t}\omega_t + \phi_t^*(\omega - \omega_0) \tag{2.16}$$

$$= \phi_t^*(\mathcal{L}_{X_t}\omega_t + \omega - \omega_0). \tag{2.17}$$

So we are reduced to finding such an X_t that makes this expression vanish:

$$\mathcal{L}_{X_t}\omega_t + \omega - \omega_0 = 0.$$

To that end, we can apply Cartan's magic formula for the first term:

$$\mathcal{L}_{X_t}\omega_t = \iota(X_t)\mathrm{d}\omega_t + \mathrm{d}\iota(X_t)\omega_t,$$

but since $\mathrm{d}\omega_t = 0$,

$$\mathcal{L}_{X_t} = \mathrm{d}\iota(X_t)\omega_t.$$

Thus we are now reduced to finding X_t satisfying

$$\mathrm{d}\iota(X_t)\omega_t = \omega_0 - \omega, \quad \forall 0 \le t \le 1. \tag{2.18}$$

This equation is called *Moser's Equation*, which need now to solve for X_t (the possibility of solving this is covered in da Silva by a single sentence on pg. 44).

By shrinking U if needed, we must have $H^*(U) = 0$ for $* > 0$. Hence there exist primitives $\lambda_0, \lambda_1 \in \Omega^1(U)$ such that

$$\mathrm{d}\lambda_0 = \omega_0, \quad \mathrm{d}\lambda_1 = \omega.$$

Explicitly, since we know

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

we could use

$$\lambda_0 = \sum_{i=1}^{n} x_i \mathrm{d}y_i.$$

So now Moser's equation becomes

$$\mathrm{d}\iota(X_t)\omega_t = \mathrm{d}\lambda_0 - \mathrm{d}\lambda.$$

To solve this, it is sufficient to solve

$$\iota(X_t)\omega_t = \lambda_0 - \lambda$$

for all t. We call this the *strong Moser's Equation.* □

Lecture 5: 27 Jan

2.4 Action

Suppose $H: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a Hamiltonian. Consider two points $z_0 = (x_0, y_0), z_1 = (x_1, y_1) \in \mathbb{R}^{2n}$, and following set:

$$\mathcal{P} = \{\text{smooth } \gamma\colon [0,1] \to \mathbb{R}^{2n} \mid \gamma(0) = z_0, \gamma(1) = z_1\}$$

which is the set of smooth paths from z_0 to z_1.

Consider the following question: Which elements of P, that is, which paths correspond to "physical paths" a particle may take in the phase space? The answer is the such a path (if one exists) must satisfy Hamilton's Equations, and the required initial conditions:

$$\gamma(0) = z_0, \gamma(1) = z_1$$

and

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma(t).$$

Lagrange's idea: Define a functional on P so that the critical points of this functional correspond to such physical paths. This functional, which depends on the Hamiltonian:

$$\mathcal{A}_H : \mathcal{P} \to \mathbb{R}$$

is called the *action*.

To define $\mathcal{A}_H$, we integrate something called the *action 1-form* λ_H along a path $\gamma(t) = (x(t), y(t))$ (the action 1-form takes a path and spits out a number).

Definition 7. The *action 1-form* is defined to be

$$\lambda_H := -\lambda_0 - H\mathrm{d}t$$

where λ_0 is as defined in the proof of Darboux's Theorem:

$$\lambda_0 = \sum_{i=1}^{n} x_i \mathrm{d}y_i$$

which is a primitive of ω_0.

Integrating the action 1-form along a path $\gamma \in P$, we get

$$\begin{aligned}
\mathcal{A}(\gamma) &= \int_\gamma \lambda_H && (2.19)\\
&= \int_0^1 \gamma^* \lambda_H && (2.20)\\
&= \int_0^1 \left[-\gamma^* \left(\sum_{i=1}^n x_i \mathrm{d}y_i \right) - \gamma^*(H\mathrm{d}t) \right] && (2.21)\\
&= \int_0^1 [-x(t) \cdot \mathrm{d}y(t) - H \circ \gamma(t)\mathrm{d}t] && (2.22)\\
&= -\int_0^1 \left(x(t) \frac{\mathrm{d}y}{\mathrm{d}t}(t) + H \circ \gamma(t) \right) \mathrm{d}t. && (2.23)
\end{aligned}$$

Proposition 2. The critical points of the action functional $\mathcal{A}_H$ are exactly the paths obeying Hamilton's Equations. In other words, a path $\gamma \in \mathcal{P}$ is a critical point of $\mathcal{A}_H$ if and only if γ satisfies Hamilton's equations.

This statement, or some variation thereof, is called *Lagrange's Principle of Least Action*. In some sense it says the path of least-action is the physical path.

Remark. We might actually have critical points which are not minimums. In fact, if we do not assume M to be closed then there may not exist any minima.

The proof of the above proposition will involve variational techniques.

Lecture 6: 30 Jan

We will first work in $\mathbb{R}^{2n}$. We fix $H \in C^\infty(\mathbb{R}^{2n})$. Fix two points $z_0 = (x_0, y_0), z_1 = (x_1, y_1) \in \mathbb{R}^{2n}$. Our path space will be

$$\mathcal{P} = \{\text{smooth } \gamma \colon [0,1] \to \mathbb{R}^{2n} \mid \gamma(0) = z_0, \gamma(1) = z_1\}.$$

Then following the previous definition, the actiona functional is

$$\begin{aligned}
\mathcal{A} = \mathcal{A}_H \colon \mathcal{P} &\to \mathbb{R}\\
\gamma &\mapsto \mathcal{A}_H(\gamma) = \int_0^1 \gamma^* \lambda_H
\end{aligned}$$

where

$$\lambda_H = \lambda_0 - H\mathrm{d}t$$

is the action 1-form, and where

$$\lambda_0 = \sum_{i=1}^n y_i \mathrm{d}x_i$$

is a primitive of

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

We also previously saw the following expression for the action functional:

$$\mathcal{A}_H(\gamma) = \int_0^1 \left(y(t) \cdot \frac{\mathrm{d}x}{\mathrm{d}t}(t) - H \circ \gamma(t) \right) \mathrm{d}t.$$

Now for the proof of the Principle of Least Action (on $\mathbb{R}^{2n}$).

Proof (Principle of Least Action, $\mathbb{R}^{2n}$ version). Fix any $\gamma \in \mathcal{P}$ and vary it in a smooth family

$$\gamma_s \colon [0,1] \to \mathbb{R}^{2n}$$

where

$$\gamma_s(0) = z_0, \quad \gamma_s(1) = z_1 \quad \forall s \in (-\varepsilon, \varepsilon)$$

for some $\varepsilon > 0$; and further,

$$\gamma_0 = \gamma.$$

Pictorially, we vary γ in both directions while holding the endpoints fixed. See Figure 2.2.

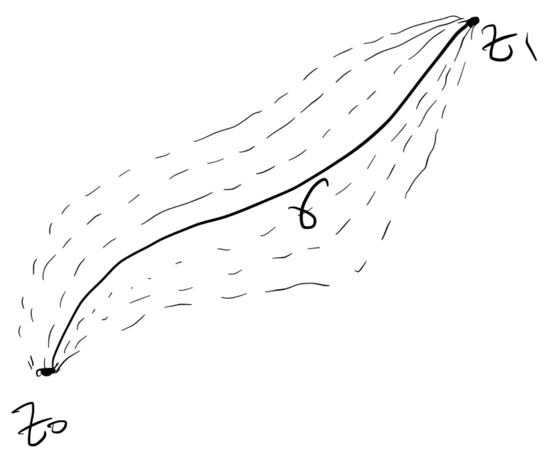

Figure 2.2: Varying a path.

By definition, γ is a critical point of $\mathcal{A}_H$ if and only if

$$\left. \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{A}_H(\gamma_s) \right|_{s=0} = 0. \tag{2.24}$$

Let us define a vector field on γ

$$\hat{\gamma}(t) = (\hat{x}(t), \hat{y}(t)) := \left.\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}s}\right|_{s=0}.$$

That is, at each point $\gamma(t)$ on γ, we have a vector pointing in the direction of how the γ_s family is changing, for that same t. One should think of this vector field as some sort of "tangent vector" to γ in $\mathcal{P}$ (this point will be elaborated on later) Pictorially:

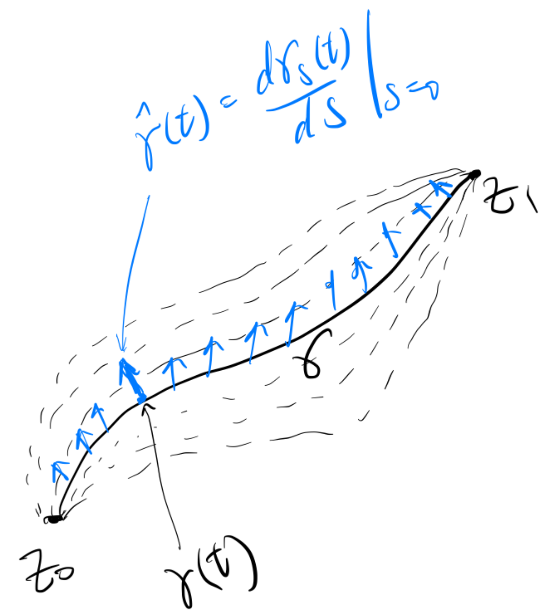

Figure 2.3: The vector field$\hat{\gamma}(t)$.

Now,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_H(\gamma_s)\right|_{s=0} \overset{\text{Fubini}}{=} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(y_s(t)\cdot\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}t} - H\circ\gamma_s(t)\right)\mathrm{d}t \tag{2.25}$$

$$= \int_0^1 \underbrace{\left(\hat{y}(H)\cdot\frac{\mathrm{d}\gamma_s(t)}{\mathrm{d}t} + y(t)\cdot\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}\right)}_{\text{product rule}}\mathrm{d}t \tag{2.26}$$

$$+ \int_0^1 \left(-\frac{\partial H}{\partial x}\cdot\hat{x}(t) - \frac{\partial H}{\partial y}\cdot\hat{y}(t)\right)\mathrm{d}t \tag{2.27}$$

Let us examine the second term in the first integral:

$$\int_0^1 y(t)\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t}\mathrm{d}t \overset{\text{by parts}}{=} y(t)\hat{x}(t)\Big|_0^1 - \int_0^1 \frac{\mathrm{d}y(t)}{\mathrm{d}t}\hat{x}(t)\mathrm{d}t.$$

Thus, after gathering the $\hat{x}(t)$ and $\hat{y}(t)$ terms, equation 2.24 is equivalent to

$$0 = \int_0^1 \left[\hat{x}(t)\left(-\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\partial H}{\partial x}\right) + \hat{y}(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\partial H}{\partial y}\right)\right]. \tag{2.28}$$

Thus γ is a critical point of $\mathcal{A}_H$ if and only if the above equation holds.

Note, only $\hat{x}$ and $\hat{y}$ depend on the variation we chose. But the above equation must hold for any variation. In particular, we can choose $\hat{x}$ to be any bump function you like, and $\hat{y} \equiv 0$ (or vice-versa).

Therefore,

$$\frac{\partial H}{\partial y} = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \frac{\partial H}{\partial x} = -\frac{\mathrm{d}y}{\mathrm{d}t}$$

as desired. □

Now we would like to generalize to the setting of general symplectic manifolds. In our proof for the $\mathbb{R}^{2n}$ case, we make use of the existence of λ_0, a primitive of ω_0. So for the first pass of Principle of Least Action on a symplectic manifold (M, ω), we will assume M is *exact*, i.e. ω has a primitive.

Lecture 7: 1st Feb

The setting is a symplectic manifold (M, ω), with the added assumption of exactness i.e. $\omega = \mathrm{d}\lambda$. In other words, we assume ω has a primitive. We also have $H \in C^\infty(M)$. Now fix points $z_0, z_1 \in M$ and consider the space

$$\mathcal{P} := \{\text{smooth } \gamma\colon [0,1] \to M \mid \gamma(0) = z_0, \gamma(1) = z_1\}$$

of paths from z_0 to z_1.

We are interested in picking out a $\gamma \in P$ such that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma(t)$$

i.e. the Hamiltonian equations are satisfied on γ.

To that end, let us define the *action*

$$\mathcal{A} = \mathcal{A}_H : \mathcal{P} \to \mathbb{R}$$

by way of first defining the *action 1-form*

$$\lambda + H\mathrm{d}t \in \Omega^1(M)$$

then the action $\mathcal{A}$ takes some $\gamma \in P$, first pulling the above 1-form back by γ, and integrate from 0 to 1. That is,

$$\mathcal{A}(\gamma) = \int \gamma^*\lambda + H \circ \gamma(t)\mathrm{d}t.$$

Proposition 3 (Principle of Least Action, exact version)**.** Suppose (M, ω) is an exact symplectic manifold. A path $\gamma \in P$ is a critical point of $\mathcal{A}_H$ if and only if Hamilton's equations are satisfied for γ:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_H \circ \gamma(t).$$

Proof. Notice first that $\gamma \in \mathrm{Crit}(\mathcal{A})$ if and only if $\mathrm{d}\mathcal{A}_\gamma = 0$ (definition of critical point). In other words, γ is a critical point if and only if

$$(\mathrm{d}\mathcal{A}_\gamma)(Y) = 0$$

for every $Y \in T_\gamma P$. But what exactly is a "tangent vector at γ inside the space P"? It should be a smooth vector field along γ.

More precisely, $Y \in T_\gamma P$ can be written as

$$Y = Y_t, \quad 0 \le t \le 1, \quad Y_t \in T_{\gamma(t)}M.$$

Now comes $\mathrm{d}\mathcal{A}(Y)$. It is rekonned as follows: Choose a variation

$$\gamma_s \in P, \quad -\varepsilon < s < \varepsilon, \quad \gamma_0 = \gamma$$

such that

$$\frac{\mathrm{d}\gamma_s}{\mathrm{d}t}(t) = Y_t.$$

In other words, the vectors in the vector field Y give the directions for the variation.

Then we can define

$$\mathrm{d}\mathcal{A}(Y) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}(\gamma_s)\bigg|_{s=0}.$$

Opening this up, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}(\gamma_s)\bigg|_{s=0} = \int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(\gamma_s^*\lambda)\bigg|_{s=0}}_{\text{first integrand}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}s}(H \circ \gamma_s(t))\bigg|_{s=0}}_{\text{second integrand}} \mathrm{d}t \tag{2.29}$$

Let us investigate the two integrands separately.

First integrand:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\gamma_s^*\lambda)\bigg|_{s=0} \overset{\text{Lee derivative}}{=} \gamma^*\mathcal{L}_Y \tag{2.30}$$

$$\overset{\text{Cartan}}{=} \gamma^*(\underbrace{\mathrm{d}(\iota(Y)\lambda)}_{1.} + \underbrace{\iota(Y)\mathrm{d}\lambda}_{2.}) \tag{2.31}$$

We have

1. :

$$\gamma^*(\mathrm{d}(\iota(Y)\lambda) = \mathrm{d}(\gamma^*(\iota(Y)\lambda)) \tag{2.32}$$
$$= \mathrm{d}(\gamma^*\lambda(Y)) \tag{2.33}$$
$$= \mathrm{d}(\lambda(Y_t))\,. \tag{2.34}$$

Hence when we evaluate the integral,

$$\int_0^1 \mathrm{d}(\lambda(Y_t)) \overset{\text{FTC}}{=} \lambda(Y_1) - \lambda(Y_0) = 0.$$

2. :

$$\gamma^*(\iota(Y)\mathrm{d}\lambda) = \gamma^*(\iota(Y)\omega) \tag{2.35}$$

Now this thing on the right hand side is a 1-form on $[0,1]$. To figure out what it is (how it acts), we plug in the constant vector field $\frac{\partial}{\partial t}$ on $[0,1]$:

$$\gamma^*(\iota(Y)\omega)\left(\frac{\partial}{\partial t}\right) = \iota(Y)\omega\left(\mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right)\right) \tag{2.36}$$
$$= \iota(Y)\omega\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \tag{2.37}$$
$$= \omega\left(Y, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right). \tag{2.38}$$

Hence $\gamma^*(\iota(Y)\omega)$ is the same 1-form on $[0,1]$ as

$$\omega\left(Y, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)\mathrm{d}t.$$

So now when we evaluate the integral:

$$\int 2. = \int_0^1 \omega\left(Y_t, \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right)\mathrm{d}t.$$

Now let us look at the second integrand.

Second integrand:

$$\frac{\mathrm{d}}{\mathrm{d}s}H\circ\gamma_s(t)\bigg|_{s=0} = \mathrm{d}H(Y_t) \tag{2.39}$$
$$= \iota(X_H)\omega(Y_t) \tag{2.40}$$
$$= \omega(X_H\circ\gamma(t), Y_t). \tag{2.41}$$

Putting the integrands together, we have

$$\mathrm{d}\mathcal{A}(Y) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}(\gamma_s)\bigg|_{s=0} \tag{2.42}$$

$$= \int_0^1 \left[\omega\left(Y_t, \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right) + \omega(X_H \circ \gamma(t), Y_t)\right] \mathrm{d}t \tag{2.43}$$

$$= \int_0^1 \omega\left(Y_t, \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - X_H \circ \gamma(t)\right) \mathrm{d}t. \tag{2.44}$$

So given any $\gamma \in P$, and a "tangent vector" $Y_t \in T_\gamma P$ (a vector field on γ), we have this above formula.

Now it follows that $\mathrm{d}\mathcal{A}(Y) \equiv 0$ if and only if the RHS of this formula vanishes for all choices of tangent vector $Y = Y_t$.

Now to establish the desired result, suppose γ satisfies Hamilton's equations:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_H \circ \gamma(t).$$

Then the formula does indeed vanish for all $Y = Y_t$. Indeed, if not, we can use the non-degeneracy of ω to build a vector field Y such that the RHS is non-zero. □

Remark. The same proof goes through if we replaced $H \in C^\infty(M)$ by a time-dependent

$$H = H_t \in C^\infty(M \times [0,1]).$$

We shall like to work with actions in more general symplectic manifolds, not just ones that are exact. We will be interested also in studying loops

$$\gamma\colon [0,1] \to M, \quad \gamma(0) = \gamma(1)$$

and such that they satisfy Hamilton's equaitons

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_{H_t} \circ \gamma(t).$$

And we shall study the space

$$\mathcal{L} := \left\{\text{smooth contractible loops}\gamma\colon S^1 \to M\right\}.$$

Here contractible can be understood as the following: for each loop $\gamma \in \mathcal{L}$, there exists a $\hat{\gamma}\colon D^2 \to M$ such that $\hat{\gamma}\big|_{\partial D^2} = \gamma$. We then define the action on loop space to be

$$\mathcal{A} := \mathcal{A}_{H_t}\colon \mathcal{L} \to \mathbb{R}$$

$$\mathcal{A}(\gamma) \int_0^1 H_t \circ \gamma(t)\mathrm{d}t \int_{D^2} \hat{\gamma}\omega.$$

Lecture 8: 3rd February

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The setting now is (M, ω), any symplectic manifold (no longer assuming exact). We consider time-dependent Hamitonians

$$H_t \in C^\infty(M \times [0,1]),$$

and define the space of smooth contractible closed loops:

$$\mathcal{L}(M) = \{\gamma \colon S^1 \to M \colon \gamma \text{ contractible}\}.$$

As mentioned before, the contractibility condition is that for any loop $\gamma \in \mathcal{L}(M)$, we can always "fill it in with a disk": there exists $\hat{\gamma} \colon D^2 \to M$ such that

$$\hat{\gamma}\Big|_{\partial D^2} = \gamma.$$

Now we define the *action functional* to be

$$\mathcal{A} = \mathcal{A}_{H_t} \colon \mathcal{L}(M) \to \mathbb{R}$$

by

$$\mathcal{A}(\gamma) = \int_0^1 H_t \circ \gamma(t) \mathrm{d}t + \int_{D^2} \hat{\gamma}^* \omega.$$

The interpretation for this formula is that of computing the "symplectic area of the disk".

> **Remark.** In the previous setting, we assumed (M, ω) is exact, i.e. $\omega = \mathrm{d}\lambda$ for some $\lambda \in \Omega^1(M)$. If we were in that setting, then
>
> $$\begin{aligned} \int_{D^2} \hat{\gamma}^* \omega &= \int_{D^2} \hat{\gamma}^* \mathrm{d}\lambda && (2.45) \\ &= \int_{D^2} \mathrm{d}(\hat{\gamma}^* \lambda) && (2.46) \\ &\overset{\text{Stokes}}{=} \int_{\partial D^2} \gamma^* \lambda && (2.47) \\ &= \int_{S^1} \gamma^* \lambda && (2.48) \\ &= \int_0^1 \gamma^* \lambda. && (2.49) \end{aligned}$$

We need to make sure that the choice of $\hat{\gamma}$ does not matter so that $\mathcal{A}$ is actually well-defined. That is,

$$\int_{D^2} \hat{\gamma}^* \omega$$

is independent of the choice of the "cappping" $\hat{\gamma}$ of γ.

To that end we will need an assumption on M to guarantee this. The assumption that will work is to assume

$$\pi_2(M) = 0.$$

Here's why: Suppose we have two cappings $\hat{\gamma}_1$ and $\hat{\gamma}_2$, then we can glue them together to get $\hat{\gamma}\colon S^2 \to M$.

Now we want to show

$$0 = \int_{D^2} \hat{\gamma}_1^* \omega - \int_{D^2} \hat{\gamma}_2^* \omega = \int_{S^2} \hat{\gamma}^* \omega.$$

The latter of which is the symplectic area of the 2-sphere. If $\pi_2(M) = 0$, then there exists and extension of $\hat{\gamma}$:

$$\hat{\hat{\gamma}}\colon B^3 \to M$$

such that

$$\hat{\hat{\gamma}}\Big|_{\partial B^3} = \hat{\gamma}.$$

Thus,

$$\int_{S^2} \hat{\gamma}^* \omega = \int_{\partial B^3} \hat{\hat{\gamma}}^* \omega \tag{2.50}$$

$$\overset{\text{Stokes}}{=} \int_{B^3} \mathrm{d}\left(\hat{\hat{\gamma}}^* \omega\right) \tag{2.51}$$

$$= \int_{B^3} \hat{\hat{\gamma}}^* \mathrm{d}\omega \tag{2.52}$$

$$\overset{\omega \text{ closed}}{=} 0. \tag{2.53}$$

So we have the desired result.

More generally, we could assume that

$$[\omega] \in H^2(M)$$

vanishes on the image of the Hurwicz homomorphism

$$\pi_2(M) \to H_2(M).$$

This condition is called *symplectically apherical.* i.e. the symplectic area of every 2-sphere vanishes.

Remark. Recall: The Huriwicz homomorphism

$$h_*\colon \pi_n(X) \to H_n(X)$$

is defined as such: Elements of $\pi_n(X)$ are maps $S^n \to X$. Choose a canonical generator $u_n \in H_n(S^n)$ (because S^n oriented to top dimensional homology is rank 1). Then an element $f \in \pi_n(X)$ is mapped to $f_*(u_n) \in H_n(X)$.

Proposition 4. If (M, ω) is symplectically aspherical, then

$$\mathcal{A} = \mathcal{A}_{H_t}\colon \mathcal{L}(M) \to \mathbb{R}$$

is well-defined for any $H_t \in C^\infty(M \times [0,1])$.

So now we are ready to state the most general form of the Principle of Least Action:

Proposition 5 (Principle of Least Action, general). If $\gamma \in \mathcal{L}(M)$, then $\gamma \in \operatorname{Crit}(\mathcal{A}_{H_t})$ if and only if

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_{H_t} \circ \gamma(t), \quad \forall t.$$

That is, $\gamma \in \operatorname{Crit}(\mathcal{A}_{H_t})$ if and only if γ is a closed orbit of the Hamiltonian flow $\phi^t_{H_t}$ (closed orbit because γ is a loop).

Proof. See reference: Audin-Damian, Morse Theory and Floer Homology. □

Why time-dependent Hamiltonians? In other words, why do we care about closed orbits of Hamiltonian flow of a time-dependent Hamiltonian? Suppose H is instead just a time-independent (sometimes called "autonomous") Hamiltonian. Then closed orbits of ϕ^1_H are recurrent states; they are in one-to-one correspondence with points $\gamma(0) \in M$ which return to their original position after time-1 flow of H. Given any (M, ω), must there be lots of recurrent states for a given H? Yes.

$$\{\text{closed orbits of } \phi^t_H, 0 \leq t \leq 1\} \longleftrightarrow \operatorname{Fix}(\phi^1_H).$$

Compare this with the statement of Lefshetz Fixed Point Theorem: If $f \in \operatorname{Diff}(M)$ for any smooth manifold M, then

$$\# \operatorname{Fix}(f) \geq |\chi(M)|.$$

Also compare with the statement of the Poincare-Birkhoff Theorem: If $f \in \operatorname{Diff}(S^1 \times [0,1])$ (this space is the annulus, which has Euler charactersitic 0). Suppose f preserves area, and rotates boundary components in opposite directions, then f gas ≥ 2 fixed points.

The correct generalization of this Theorem to higher dimensions is not to require "volume preserving", but rather to require "preserving the symplectic form" (but this raises the obvious question: what about odd-dimensional manifolds?).

2.5 Arnold Conecture

Lecture 9: 8 Feb

Setting: (M, ω) closed symplectic manifold. We want a Lie subgroup $H \leq \operatorname{Symp}(M, \omega)$ (symplectomorphisms of (M, ω)).

Observation 1: Given a family $\phi_t \in \operatorname{Symp}_0(M, \omega)$ such that $\phi_0 = \operatorname{Id}$. This is in one-to-one correspondence with $\alpha_t \in \Omega^1(M), 0 \leq t \leq 1$, α_t closed (both smooth in t). Reminder: $\Omega^1(M)$ are 1-forms, i.e. they take a single vector from the tangent space and spit out a number. Closed means $\mathrm{d}\alpha_t = 0$.

Proof. Suppose we have such a family $\phi_t \in \mathrm{Symp}_0(M,\omega)$ of symplectomorphisms with the added condition of $\phi_0 = \mathrm{Id}$ is the identity diffeomorphism. By the definition of symplectomorphism,

$$\phi_t^*\omega = \omega, \quad 0 \le t \le 1$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\omega = 0$$

and

$$\phi_0 = \mathrm{Id}$$

if and only if

$$\mathcal{L}_{X_t}\omega = 0$$

where X_t is the time-dependent vector field generating ϕ_t. This means the following (pg. 35 of da Silva): Whenever there is an isotopy $\phi = \phi_t \colon M \to M$ (through diffeomorphisms), there is an associated family of vector fields X_t, which at each $p \in M$ satisfy

$$X_t(p) = \frac{\mathrm{d}}{\mathrm{d}s}\phi_s(\phi_t^{-1}(p))\bigg|_{s=t}.$$

This jumble of symbols really just says at time t, the vector at a point should point in the direction in which ϕ is changing at time t.

Now back to the proof.

In this case, Cartans formula

$$\mathcal{L}_{X_t}\omega = \iota(X_t)\mathrm{d}\omega + \mathrm{d}\iota(X_t)\omega$$

reduces to

$$0 = \mathrm{d}\iota(X_t)\omega$$

because ω is a closed form. Hence the family of 1-forms α_t defined as

$$\alpha_t = \iota(X_t)\omega$$

is closed for $0 \le t \le 1$.

This is the correspondence

$$\{\phi_t\} \to \{\alpha_t\}$$

Other direction: Given $\alpha + t$. There exists a time-dependent vector field X_t such that $\iota(X_t)\omega = \alpha_t$. Using nondegeneracy of ω.

Because M is closed, we can integrate X_t for time 1 to get isotopy ϕ_t, $0 \le t \le 1$ with $\phi_0 = \mathrm{Id}$.

Conclude that $\phi^*\omega = \omega$ $0 \le t \le 1$ using Cartans firmula and the Lie derivative. □

Observation 2: 1-parameter subgroups of $\mathrm{Symp}(M,\omega)$ are in one-to-one correspondence with closed 1-forms on M. Why is this:

Proof. Given α a closed 1-form. Define $\alpha_t = \alpha$ for all $t \in \mathbb{R}$ the constant 1-parameter family. Then get time-indenedent vecor field X and time-t flow $\phi_X^t = \phi_H^t$, where $X = X_H$.

Note: Because X is autonomous, $\phi_H^t \circ \phi_H^s = \phi_H^{t+s}$.

Conversely, given $\{\phi^t\}$ a 1-parameter subgroup of $\mathrm{Symp}(M, \omega)$, march through the proof of Observation 1 (forward direction) to get α_t closed 1-forms for all $t \in \mathbb{R}$. Check that α_t is independent of t.

Suppose that $\alpha = \mathrm{d}H$ is exact. Then the associated vector field X satisfies

$$\alpha = \iota(X)\omega = \mathrm{d}H.$$

Consider $\phi = \phi_H^1$. The fixed points

$$\mathrm{Fix}(\phi) \supset \{p \in M \colon X_p = 0\} \tag{2.54}$$
$$= \{p \in M \colon \mathrm{d}H_p = 0\} \tag{2.55}$$
$$= \mathrm{Crit}(H). \tag{2.56}$$

Hence

$$\#\,\mathrm{Fix}(\phi) \geq \min\{\#\,\mathrm{Crit}(H) \colon H \in C^\infty(M)\}$$

By contrast if we choose α to be closed but not exact, then ϕ may not have fixed points at all.

Example. $(M, \omega) = (T^2, \omega_0)$. We have angular coordinates (θ, ψ). Consider the closed form $\mathrm{d}\theta$. It is not exact.

We have $X = \frac{\partial}{\partial\theta}$. Then ϕ_H^t has no fixed points for $0 < t < 2\pi$.

A sensible candidate for $H \leq \mathrm{Symp}_0(M, \omega)$ is the smallest subgroup containing

$$\{\phi_H^1 \colon H \in C^\infty(H)\}.$$

Here the notation ϕ_H^1 means the time-1 symplectomorphism of a family of Hamiltonian isotopies. A Hamiltonian isotopy is a symplectic isotopy wherein each $\alpha_i = \mathrm{d}H_i$ is exact. The problem of estimating

$$\min\{\#\,\mathrm{Crit}(H) \colon H \in C^\infty(M)\}$$

is Lyusternik-Schnirelman Theory. Define the *cup-length* $cl(M)$ of M to be the maximum integer k such that there exists $\alpha_1, \ldots, \alpha_k \in H^*(M)$, $|\alpha_i| \geq 1$, such that

$$\alpha_1 \smile \cdots \smile \alpha_k \neq 0.$$

Then

$$cl(M) \leq \dim(M).$$

For example, $M = T^2$ has $cl(M) = \dim(M)$.

Theorem 5.

$$cl(M) + 1 \leq \min\{\#\,\mathrm{Crit}(H) \colon H \in C^\infty(M)\}$$

Example.

$$k + 1 \leq \dots M = T^k$$

Check that this bound is attained.

Hence if $H \in C^\infty(T^{2n})$, then $\#\operatorname{Fix}(\phi_H^1) \geq 2n + 1$.

Going back....

Note: if $H, G \in C^\infty(M)$, then in general

$$\phi_H^1 \circ \phi_G^1$$

may not be equal to

$$\phi_K^1$$

for some $K \in C^\infty(M)$. However, if we define

$$K_t \in C^\infty(M \times [0, 1])$$

by

$$K_t = H + G \circ \phi_H^t,$$

then it is an exercise to check

$$\phi_{K_t}^t = \phi_H^t \circ \phi_G^t$$

in particular it is true that

$$\phi_{K_t}^1 = \phi_H^1 \circ \phi_G^1.$$

Moreover if in the beginning we had H_t, G_t, and we went rhough to define

$$K_t = H_t + G_t \circ \phi_H^t$$

then

$$\phi_{K_t}^t = \phi_{H_t}^t \circ \phi_{G_t}^t$$

in particular it is true that

$$\phi_{K_t}^1 = \phi_{H_t}^1 \circ \phi_{G_t}^1.$$

Check, if we define instead:

$$K_t \coloneqq -H_t \circ \phi_{H_t}^t$$

then

$$\phi_{K_t}^t = {\phi_{H_t}^t}^{-1}$$

Thus

$$H = \operatorname{Ham}(M, \omega) \coloneqq \{\phi_{H_t}^1 : H_t \in C^\infty(M \times [0, 1]), \alpha_t = \mathrm{d}H_t\} \trianglelefteq \operatorname{Symp}_0(M, \omega)$$

is an honest subgroup of $\operatorname{Symp}(M, \omega)$. Elements of $\operatorname{Ham}(M, \omega)$ are called *Hamiltonian symplectomorphisms*. What the definition is say is f is a Hamiltonian symplectomorphism if there exists a Hamiltonian isotopy $\{\phi_{H_t}^t\}$ such that $f = \phi_{H_t}^1$.

Indeed, Arnold conjectured that if $\phi \in \operatorname{Ham}(M, \omega)$, then

$$\#\operatorname{Fix}(\phi) \geq \min\{\#\operatorname{Crit}(H) : H \in C^\infty(M)\}.$$

(The RHS says take minimum over *all* arbitrary functions H on M) This is obvious if $\phi = \phi_H^1$, but not obvious if $\phi = \phi_{H_t}^1$.

Motivations of Arnold:

1. Poincare-Birkhoff Theorem
2. The conjecture holds if $\phi \in \mathrm{Hom}(M, \omega)$ is chosen sufficiently close to Id.

There exists stronger conclusion if we put stronger hypothesis of $\mathrm{Fix}(\phi)$, namely that all fixed points are *non-degenerate*:

Given $\phi \in \mathrm{Diff}(M)$, where M is just some smooth manifold. We can form its graph

$$\Gamma(\phi) := \{(p, \phi(p)) : p \in M\} \subset M \times M.$$

A special case is if $\phi = \mathrm{Id}$, then we write $\Gamma(\phi) = \Delta$, the diagonal.

Thus,

$$\mathrm{Fix}(\phi) \leftrightarrow \Gamma(\phi) \cap \Delta$$

wehre

$$p \mapsto (p, p).$$

Both Δ and $\Gamma(\phi)$ are submanifolds of $M \times M$ diffeo to M.

A fixed point $p \in \mathrm{Fix}(\phi)$ is said to be *non-degenerate* if (p, p) is a transverse point of intersection of $\Gamma(\phi)$ and Δ. Hence all fixed points $\mathrm{Fix}(\phi)$ consists of non-degenerate fixed points if and only if

$$\Gamma(\phi) \pitchfork \Delta.$$

Arnold Conjecture #2: If $\phi \in \mathrm{Hom}(M, \omega)$ and all fixed points of ϕ are non-degenerate, then

$$\# \mathrm{Fix}(\phi) \geq \min\{\# \mathrm{Crit}(H) : H \text{ a Morse function on } M\}.$$

Chapter 3

Morse Theory

Lecture 10: 10 Feb

Motivations are conjectures of Arnold. Suppose (M, ω) is a closed symplectic manifold. Suppose $\phi \in \mathrm{Ham}(M, \omega)$, i.e.

$$\phi = \phi^1_{H_t}, \quad H_t \in C^\infty(M \times [0,1]).$$

As previously seen. Arnold Conjectures:

1. $\#\,\mathrm{Fix}(\phi) \geq \min\{\#\,\mathrm{Crit}(f) \colon f \in C^\infty(M)\}$. (The RHS is saying the minimum over *all* arbitrary functions on M).
2. If $\mathrm{Fix}(\phi)$ consists of non-degenerate fixed points (defined previously), then

$$\#\,\mathrm{Fix}(\phi) \geq \min\{\#\,\mathrm{Crit}(f) \colon f \text{ a Morse function on } M\}.$$

Remark. Suppose $H \in C^\infty(M)$ (thinking about it as a Hamiltonian), then $\phi^1_{\varepsilon \cdot H} = \phi^\varepsilon_H$. Also,

$$\mathrm{Crit}(H) = \mathrm{Crit}(\varepsilon \cdot H).$$

For small ε,

$$\mathrm{Fix}(\phi^1_{\varepsilon H}) = \mathrm{Fix}(\phi^\varepsilon_H) = \mathrm{Crit}(H).$$

This is true locally, and true globally when the manifold is compact.

This is an evidence for the first Arnold conjecture.

Out-of-order remark: If $\mathrm{Fix}(\phi^1_{\varepsilon H})$ are non-degenerate, then H is a Morse function, and we get evidence for the second conjecture.

3.1 Morse Functions

Suppose M is a closed, smooth manifold and $f \in C^\infty(M)$. Suppose $p \in \mathrm{Crit}(f)$ i.e. $\mathrm{d}f_p = 0$.

Non-degeneracy of critical point (the following is a coordinate-free way of defining this concept, some sources define it otherwise) We can consider the *Hessian* of (f,p),

$$H = \mathrm{Hess}(f,p)\colon T_pM \times T_pM \to \mathbb{R}$$

a symmetric bilinear pairing. Then we will say p is (non)-degenerate according to whether H is (non)degenerate.

We need to specify what $H(X_p, Y_p)$ is for $X_p, Y_p \in T_pM$. First we extend both X_p, Y_p to vector fields X, Y in a neighborhood of p. Consider the action of X on f, $X \cdot f$, in this neighborhood. That is, the directional derivative of f in the direction of X. So for some point q in the neighborhood U,

$$X \cdot f(q) = \mathrm{d}f_q(X_q) \in \mathbb{R}.$$

So $X \cdot f$ is a real valued function on U.

We can also consider $Y \cdot (X \cdot f)$, again a real-valued function on U, which we will use to define:

$$H(X_p, Y_p) \coloneqq Y \cdot (X \cdot f)(p).$$

Now why is $H(X_p, Y_p)$ independent of the choice of X and Y (the extensions), i.e. why is it tensorial; and why is it symmetric in its arguments?

First, note that

$$Y \cdot (X \cdot f)(p) = \mathrm{d}(X \cdot f)_p\,(Y_p),$$

so $H(X_p, Y_p)$ is at least independent of the choice of extension of Y (here p is the specific p we started off with, not any arbitrary point in U; thus Y_p is always the same for any extension). Similarly, if we define

$$H'(X_p, Y_p) = X \cdot (Y \cdot f)(p)$$

is independent of the choice of X. Thus,

$$\begin{aligned} H(X_p, Y_p) - H'(X_p, Y_p) &= Y \cdot (X \cdot f)(p) - X \cdot (Y \cdot f)(p) && (3.1)\\ &= [Y, X] \cdot f(p) && (3.2)\\ &= \mathrm{d}f_p([Y, X]_p) && (3.3)\\ &= 0 \quad (p \in \mathrm{Crit}(f)). && (3.4) \end{aligned}$$

This completes the proof that H does not depend on the extension (so well-defined), and is symmetric (and bilinear).

Now $\mathrm{Hess}(f,p)$ has a signature (s, u, z) where

- s is the maximum dimension of a subspace of T_pM on which H is positive definite ($H(x,x) > 0 \quad \forall x \neq 0$). Here s stands for *stable*. Corresponding to positive eigenvalues (dimension of eigenspace assocaited to them).
- u is the maximum dimension of a subspace of T_pM on which H is negative definite. Here u stands for *unstable*. Corresponding to negative eigenvalues (dimension of eigenspace assocaited to them).
- z is the maximum dimension of a subspace of T_pM on which H is zero. Here z stands for *zero*. Corresponding to zero eigenvalues.

Then

$$s + u + z = \dim T_pM.$$

We say $\mathrm{Hess}(f,p)$ is non-degenerate if $z = 0$.

Definition 8. A critical point $p \in \mathrm{Crit}(f)$ is *non-degenerate* if $\mathrm{Hess}(f, p)$ is non-degenerate. In which case we call u the *index* of p:

$$\mathrm{ind}(p) := u.$$

Lemma 1 (Morse Lemma)**.** If $p \in \mathrm{Crit}(f)$, then there exists coordinates

$$g\colon \mathbb{R}^n \supset V \to U \subset M$$

$$0 \mapsto p$$

such that

$$(f \circ g)(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_u^2) + (x_{u+1}^2 + \cdots + x_{x+s}^2) + f(p).$$

If $n = 2$, we can draw out different local models for f. See Figure 3.1.

Definition 9. A function $f \in C^\infty(M)$ is called a *Morse function* if $\mathrm{Crit}(f)$ consists of only non-degenerate critical points.

Some basic facts:

- $\mathrm{Crit}(f)$ consists of isolated points if $f\colon M \to \mathbb{R}$ is a Morse function. Hence $\mathrm{Crit}(f)$ is finite for closed M.
- Being Morse is a *generic condition* in the sense that the space of Morse functions is a dense open set in $C^\infty(M)$ (or might be a countable intersection of these).
- Morse inequality (first pass): If f is a Morse function, then

  $$\#\,\mathrm{Crit}(f) \geq \sum_{k=1}^{n} b_k(M) = \mathrm{rank}\, H_*(M, \mathbb{Z}).$$

  Basic reason why this should hold: there is a chain complex

  $$CM_*(M, f, g)$$

  (where g is an auxiliary metric), that is freely generated by the critical points of f. In fact $CM_k(M, f, g)$ is freely generated by the index k critical points. This chain complex's homology computes $H_*(M, \mathbb{Z})$.

Lecture 11: 13 Feb

Continuing Morse theory. Set-up: M is a smooth closed manifold, and $f \in C^\infty(M)$, and we assume f is Morse.

The goal now is to prove the folowing Morse inequality:

$$\#\,\mathrm{Crit}(f) \geq \mathrm{rank}\, H_*(M; \mathbb{Z})$$

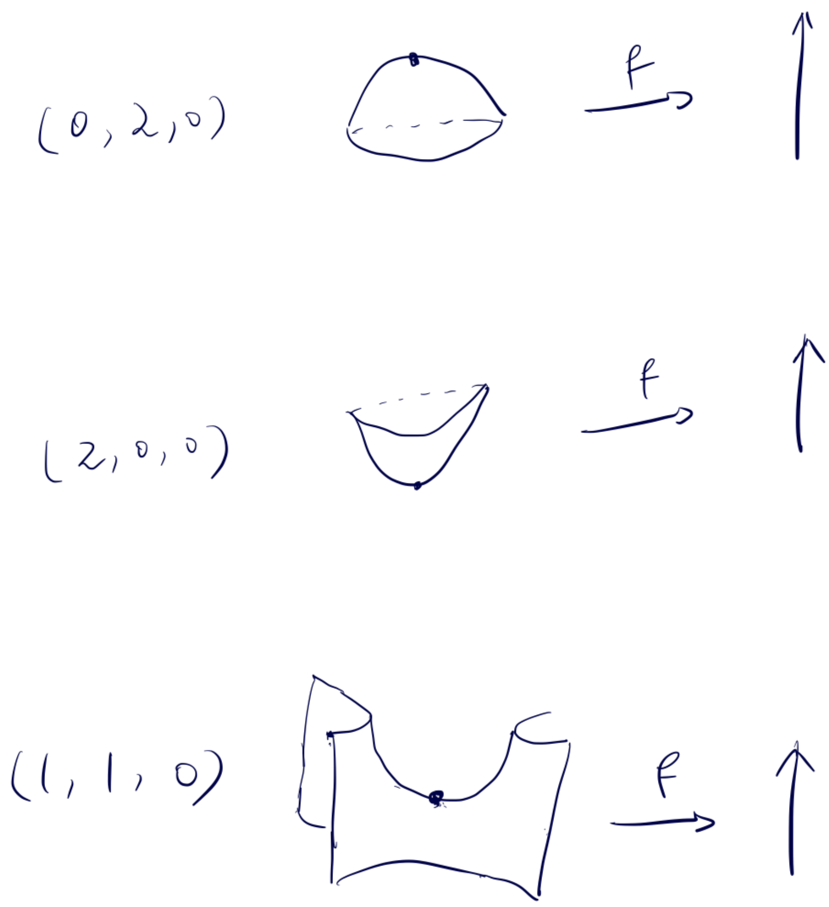

Figure 3.1: Various local models.

Basic reason why this is true: there is a chain complex $CM_*(M, f, g)$ such that $CM_k(M, f, g)$ is freely generated by index-k critical points:

$$CM_k(M, f, g) = \mathbb{Z} \cdot \langle p \rangle$$

where

$$p \in \mathrm{Crit}(f), \quad \mathrm{ind}(p) = k.$$

Further,

$$HM_*(M, f, g) \cong H_*(M, \mathbb{Z}).$$

In fact, we will find a cell decomposition of M such that we get one k-cell for each index-k critical point. Then $CM_*(M, f, g)$ is the corresponding cellular chain complex.

Recall if p is a critical point of f, there is a Hessian $\mathrm{Hess}(p, f)$ that is non-degenerate; and we let (informally) k be the number of negative eigenvalues, and $n - k$ be the number of positive eigenvalues. Here $n = \dim M$.

Recall also that Morse lemma says there exist coordinates $x_1, \ldots, x_n$ near p sucgh that

$$f(x_1, \ldots, x_n) = f(p) - (x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2).$$

To write down the boundary operator in Morse homology, or the boundary of the cells of the cell decomposition, we need an auxiliary choice of metric g. This g allows us to convert f into a vector field called the *gradient vector field of f*:

$$\nabla f = \nabla_g f.$$

This is uniquely defined by the condition

$$\mathrm{d}f = \iota(\nabla f) g,$$

just using non-degeneracy of g. Since it is an auxiliary choice, we can choose any g and for our purposes it'll be as good as any other.

Denote by ϕ_t the time-t flow of ∇f, for $t \in \mathbb{R}$. By compactness of M, it exists for all time $t \in \mathbb{R}$. Because the vector field is autonomous (time-independent), this is a one-parameter subgroup

$$\{\phi_t\} \subset \mathrm{Diff}(M).$$

The identity

$$\frac{\mathrm{d}\phi_t}{\mathrm{d}t} = (\nabla f) \circ \phi_t$$

holds for all $x \in M$.

For reasons that will be clear later, we'd like to work with $-\nabla f$.

Also,

$$\phi_{t+s} = \phi_t \circ \phi_s$$

for all $t, s \in \mathbb{R}$.

Recall that Hamiltonian flow preserves H; in contrast, here f is not preserved by the flow because f decreases under negative gradient flow:

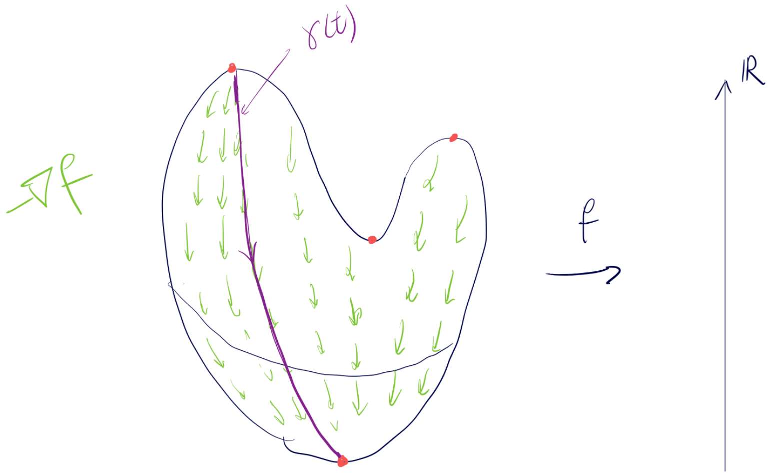

Figure 3.2: Negative gradient flow

Proposition 6. f decreases under negarive gradient flow.

Proof. Choose a flow line of $-\nabla f$, i.e. choose a differentiable map

$$\gamma \colon \mathbb{R} \to M$$

such that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = (-\nabla f) \circ \gamma(t).$$

We want to show that

$$f \circ \gamma \colon \mathbb{R} \to \mathbb{R}$$

is a decreasing function of t. To that end, consider its derivative

$$\begin{aligned}
\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}(t) &= \mathrm{d}f\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right) && (3.5)\\
&= \mathrm{d}f(-\nabla f \circ \gamma(t)) && (3.6)\\
&= \iota(\nabla f \circ \gamma(t))g(-\nabla f\gamma(t)) && (3.7)\\
&= -g(\nabla f \circ \gamma(t), \nabla f \circ \gamma(t)) && (3.8)\\
&\leq 0 \quad \text{negative-definiteness of } g && (3.9)
\end{aligned}$$

Note, equality folds if and only if

$$\nabla f \circ \gamma(t) = 0$$

i.e. $\gamma(t)$ is a critical point of f. If in fact $\gamma(t)$ is a critical point, then $\gamma(s) = \gamma(t)$ for all $s \in \mathbb{R}$, i.e. is constant. So either $\gamma \colon \mathbb{R} \to M$ is a constant map to a critical point, or else it is a non-constant flowline, and in this case,

$$\mathrm{Im}(\gamma) \cap \mathrm{Crit}(f) = \emptyset$$

and $f \circ \gamma(t)$ is strictly decreasing. □

If γ is a flowline of $-\nabla f$, then

$$\gamma(t+s) = \phi_t \circ \gamma(s).$$

Flow lines admit reparameterization: given any flow line $\gamma\colon \mathbb{R} \to M$ of ∇f,

$$\tilde{\gamma}(t) = \gamma(s+t)$$

for some fixed $s \in \mathbb{R}$, then $\tilde{\gamma}$ is a reparametrization of the same flow line.

Every point in $M - \mathrm{Crit}(f)$ lies on the image of a flow line which is unique up to reparametrization. So if $x \in M$, then there exists a unique (up to reparametrization) flow line

$$\gamma^x \colon \mathbb{R} \to M$$

of $-\nabla f$ such that $\gamma^x(0) = x$.

Now suppose γ is a flow line, we ask: what is its limiting behavior? i.e. what is

$$\lim_{t \to \pm\infty} \gamma(t)?$$

Does this limit exist?

Answer: Yes, the limit exists, and the limit is a critical point of f.

Proof. It suffices to show

$$\|\nabla f \circ \gamma(t)\|_g^2 = g(\nabla f \circ \gamma(t), \nabla f \circ \gamma(t))$$

tends to 0 as $t \to \infty$.

To show this, consider any pair of real values $a < b$. Consider

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\mathrm{d}f \circ \gamma}{\mathrm{d}t}(t)\mathrm{d}t \tag{3.10}$$

$$= -\int_a^b \|\nabla f \circ \gamma(t)\|_g^2 \mathrm{d}t \tag{3.11}$$

so

$$f(\gamma(a)) - f(\gamma(b)) = \int_a^b \|\nabla f \circ \gamma(t)\|_g^2 \mathrm{d}t \tag{3.12}$$

$$\geq (b-a) \inf_{t \in [a,b]} \|\nabla f \circ g(t)\|_g^2 \tag{3.13}$$

so

$$\frac{f(\gamma(a) - f \circ \gamma(b)}{b-a} \geq \inf_{t \in [a,b]} \|\nabla f \circ g(t)\|_g^2$$

hence

$$\frac{\max(f) - \min(f)}{b-a} \geq \inf_{t \in [a,b]} \|\nabla f \circ g(t)\|_g^2$$

Set $b = 2a$ and let $a \to \infty$ to get

$$0 = \liminf_{t\to\infty} \|\nabla f \circ g(t)\|_g^2$$

then add ε to get

$$\lim_{t\to\infty} \gamma(t) \in \mathrm{Crit}(f).$$

Definition 10. For $p \in \mathrm{Crit}(f)$, define the *stable manifold* to be

$$W_s(p) = \left\{ x \in M \colon \lim_{t\to\infty} \gamma^x(t) = p \right\},$$

and the *unstable manifold* to be

$$W_u(p) = \left\{ x \in M \colon \lim_{t\to-\infty} \gamma^x(t) = p \right\}.$$

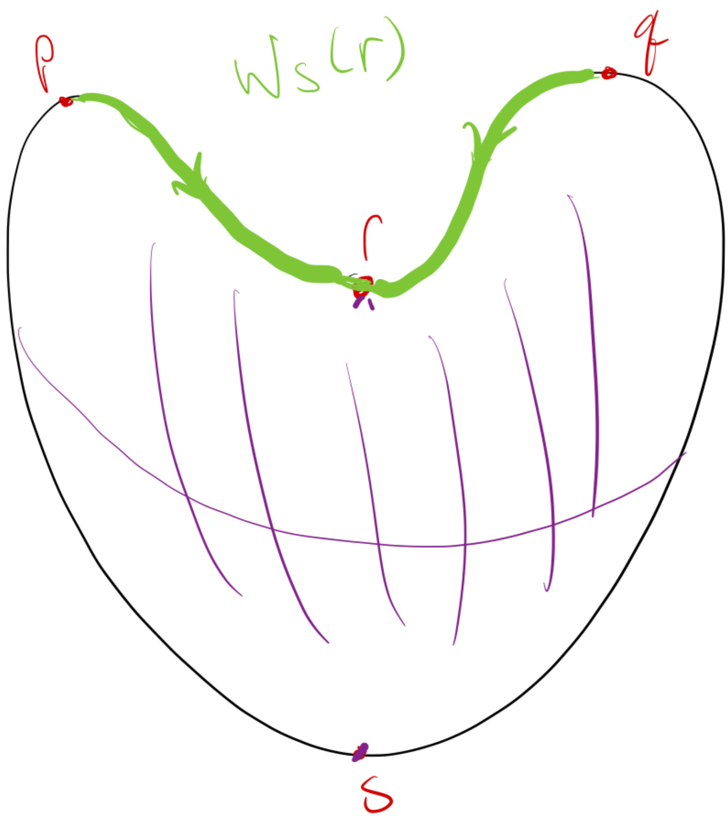

Figure 3.3: Caption

Wee see that topologically,

$$W_s(p), W_s(q) \cong \mathbb{R}^0$$

$$W_s(r) \cong \mathbb{R}^1$$

$$W_s(s) \cong \mathbb{R}^2$$

3.2 Morse Homology

Lecture 11: 15 Feb

Set up is M a smooth closed manifold, and f a Morse function on M. If g is a metric on M, we get a gradient vector field $\nabla_g f$ (depends on g). It is implicitly defined by the equation

$$\mathrm{d}f = \iota(\nabla_g f)g.$$

We can study the nagetive gradient flow of this vector field. It gives a one-parameter subgroup

$$\{\phi_t\} \subset \mathrm{Diff}(M)$$

defined as such:

$$\phi_0 = \mathrm{Id}$$

$$\frac{\mathrm{d}\phi_t}{\mathrm{d}t} = -\nabla f \circ \phi_t.$$

(This $\{\phi_t\}$ is just the flow of $-\nabla_g f$)

For all $x \in M$, flowing along the gradient flow (in either direction):

$$\lim_{t \to \pm\infty} \phi_t(x)$$

these limits exist and are critical points of f.

For each $p \in \mathrm{Crit}(f)$, we can consider its unstable and stable manifold, defined previously.

> **Example** (bumpy sphere). Morse function is height.

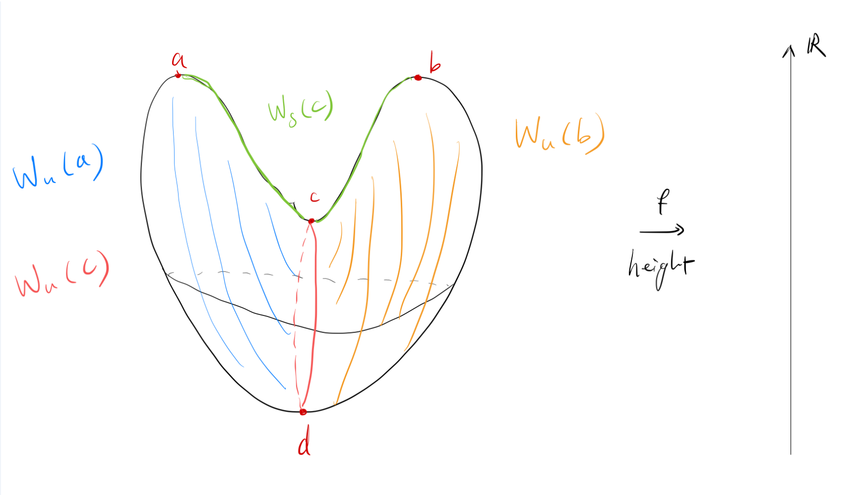

Figure 3.4: Bumpy sphere.

Example (perfectly balanced torus)**.** Morse function is height.

Theorem 6 (Thom)**.** For any $p \in \text{Crit}(f)$, the unstable manifold $W_u(p)$ is an open cell of dim $= \text{ind}(p)$; and the stable manifold $W_s(p)$ is an open cell of dim $= n - \text{ind}(p)$.

Hence there exists a diffeomorphism

$$\psi_p^\circ \colon int(B^k) \to W_u(p)$$

where $k = \text{ind}(p)$ and it extends uniquely to a smooth map

$$\psi_p \colon B^k \to M$$

because the open cell is "attached" to M.

Corollary. The collection (ψ_p, B^k) for $p \in \text{Crit}(f)$ gives a cell decomposition of M.

We can calculate $H_*(M;\mathbb{Z})$ from this cell decomposition. As a consequence, we get the Morse inequality

$$\#\,\text{Crit}(f) = \dim CM_*(M, f, g) \geq \dim H_*(M, \mathbb{Z})$$

here $CM_*(M, f, g)$ is the affiliated cellular chain complex.

Tangent spaces There are nice descriptions for the tangent spaces at p: $T_pW_u(p)$ and $T_pW_s(p)$, namely, that they intersect transversally:

$$W_u(p) \pitchfork W_s(p),$$

and they intersect precisely at p. To show this:

We have two non-degenerate pairings, the Hessian

$$\mathrm{Hess}(f,p)\colon T_pM \times T_pM \to \mathbb{R}$$

and the metric

$$g_p\colon T_pM \times T_pM \to \mathbb{R}.$$

Now $T_pW_u(p)$ is a distinguished maximal dimensional subspace of T_pM on which $\mathrm{Hess}(f,p)$ is negative definitive. Which one is it? There exist linear isomorphisms

$$\begin{aligned}\Phi_H\colon T_pM &\to T_p^*M\\ v_p &\mapsto \iota(v_p)\,\mathrm{Hess}(f,p)\end{aligned}$$

and

$$\begin{aligned}\Phi_g\colon T_pM &\to T_p^*M\\ v_p &\mapsto \iota(v_p)g_p\end{aligned}$$

Hence a linear isomoprhism

$$\Phi_g^{-1}\circ\Phi_H\colon T_pM \to T_pM$$

Exercise:

$$\begin{aligned}T_pW_u(p) &= \text{negative eigenspace of } \Phi_g^{-1}\circ\Phi_H\\ T_pW_s(p) &= \text{positive eigenspace of } \Phi_g^{-1}\circ\Phi_H\end{aligned}$$

thus

$$T_pW_u(p)\oplus T_pW_s(p) = T_pM$$

hence

$$W_u(p)\pitchfork W_s(p).$$

Issues:

1. How to see the boundary operator more explicitly?
2. We know $H_*(CM_*(M,f,g))$ is independent of the choice of f and g from cellular/singular homology. But is there a way to see this internally to Morse theory? i.e. not having to go through singular homology. The answer is yes, and this informs invariance proofs in Lagrangian Fleor homology.

To address these issues, we will define $CM_*(M,f,g)$ without reference to cellular homology. To that end, we will need a restricted choice of metric.

Definition 11. If $f\colon M\to\mathbb{R}$ is a Morse function, then a metric g on M is called *Morse-Smale* (with respect to f) if

$$W_u(p)\pitchfork W_s(q)$$

for all $p,q\in\mathrm{Crit}(f)$.

> **Note.** cf. previous discussion which only involved one critical point, whereas this definition involves two.

Observe. We can see that for the bumpy sphere, the g is Morse-Smale; however, for the torus, the g is NOT Morse-Smale because $W_u(q)$ is not transverse to $W_s(s)$.

Property. The Morse-Smale condition is generic.

Assuming g is Morse-Smale, then

$$\mathcal{M}(p,q) \coloneqq W_u(p) \cap W_s(q)$$

is a smooth manifold, typically not compact, and could possibly be empty. Its dimension, from the transversality condition, is

$$\begin{aligned} \dim W_u(p) + \dim W_s(q) - n &= \operatorname{ind}(p) + (n - \operatorname{ind}(q)) - n && (3.14) \\ &= \operatorname{ind}(p) - \operatorname{ind}(q). && (3.15) \end{aligned}$$

In fact this is a moduli space of flow lines:

$$\mathcal{M}(p,q) = \left\{ x \in M \colon \lim_{t \to -\infty} \phi_t(x) = p, \lim_{t \to \infty} \phi_t(x) = q \right\}.$$

There is an $\mathbb{R}$-action on $\mathcal{M}(p,q)$: for $t \in \mathbb{R}$, and $x \in \mathcal{M}(p,q)$,

$$t \cdot x = \phi_t(x)$$

(from x, flow for time t on its flow line).

Now define

$$\hat{\mathcal{M}}(p,q) \coloneqq \mathcal{M}(p,q) \,/\mathbb{R}$$

(thinking of this as modding out reparametrizations). It follows that

$$\dim \hat{\mathcal{M}}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1.$$

Hence if

$$\operatorname{ind}(p) = \operatorname{ind}(q) + 1$$

then $\hat{\mathcal{M}}(p,q)$ is finite set of points, and it is oriented(?). This leads us to define

$$\partial(p) = \sum_{\substack{q \in \operatorname{Crit}(f) \\ \operatorname{ind}(q) = \operatorname{ind}(q) - 1}} \# \hat{\mathcal{M}}(p,q) \cdot q.$$

Lecture 13: 17 Feb

Set up: M is a closed smooth manifold, $f \colon M \to \mathbb{R}$ a Morse function, and g a Morse-Smale metric for (M,f): for all $p,q \in \operatorname{Crit}(f)$,

$$W_u(p) \pitchfork W_s(q).$$

Hence

$$\mathcal{M}(p, q) = W_u(p) \cap W_s(q)$$

is a smooth manifold, and there is a free and smooth $\mathbb{R}$-action

$$t \cdot x = \phi_t(x)$$

for all $t \in \mathbb{R}$ and $x \in \mathcal{M}(p, q)$ (provided $p \neq q$). Dividing out the action:

$$\hat{M}(p, q) = \mathcal{M}(p, q) \,/\mathbb{R}$$

is a smooth manifold of dimension $\text{ind}(p) - \text{ind}(q) - 1$. This $\hat{\mathcal{M}}(p, q)$ is the space of unparametrized flow lines from p to Q.

If $\text{ind}(p) - \text{ind}(q) = 1$, then $\hat{\mathcal{M}}(p, q)$ consists of finitely many points ("points" are unparamterized flow lines). If we orient everything, we can think of these points having either $+$ or $-$ attached to it. In this case we can define a number

$$n(p, q) := \#\hat{\mathcal{M}}(p, q)$$

for each pair (p, q) with index difference 1. This number is an integer, and if we are careful with orientations. If not, it is well-defined mod 2.

Example (bumpy sphere)**.**

Now let $CM_*(M)$ be the free abelian group (or, if we are being lazy, the $\mathbb{F}_2$-vector space) generated by $\text{Crit}(f)$. So $CM_k(M, f, g)$ is freely generated by the $p \in \text{Crit}(f)$ where $\text{ind}(p) = k$, Now to define the differential: for $p \in \text{Crit}(f)$,

$$\partial(p) = \sum_{\substack{q \in \text{Crit}(f) \\ \text{ind}(q) = \text{ind}(q) - 1}} n(p, q) \cdot q.$$

Example.

$$\partial(a) = c, \partial(b) = c, \partial(c) = d - d = 0.$$

Theorem 7. $\partial^2 = 0$, so $(CM_*(M, f, g), \partial)$ is a chain compelx.

One proof of the Theorem is to check that ∂ matches the differential on cellular chain complex $CC_*(M, f, g)$, from the earlier point of view. However, we want a proof internal to Morse homology.

Remark. The dots-and-arrows picture of $CM_*(M, f, g,)$ exactly matches the picture of critical points and index-one flow lines on M:

pic

3.3 Poincare Duality

We would like to see Poincare Duality in Morse homology.

From the dots-and-arrows picture of the chain complex, we can directly sefine cochains

$$CM^*(M, f, g)$$

graphically, by reversing the arrows:

Then

$$H^*(M) = H_*(CM^*(M, f, g)).$$

Observe.

$$CM^*(M, f, g) = CM_*(M, -f, g).$$

And notice

$$\text{ind}(p, f) + \text{ind}(p, -f) = n$$

since

$$\text{ind}(p, f) = \dim W_u(p, f)$$

and

$$\text{ind}(p, -f) = \dim W_u(p, -f) = \dim W_s(p, f).$$

Consequently,

$$H^*(M) \cong H_{n-*}(M).$$

Hence Poincare Duality comes from turning f upside down.

3.4 Why $\partial^2 = 0$

Fix a creitical point $p \in \text{Crit}(f)$. Then

$$\partial^2(p) = \sum_{\substack{q \in \text{Crit} \\ \text{ind}(q) = \text{ind}(p) - 1}} n(p, q) \cdot \partial q \tag{3.16}$$

$$= \sum_{\substack{q \in \text{Crit}(f) \\ \text{ind}(q) = \text{ind}(p) - 1}} \sum_{\substack{r \in \text{Crit}(f) \\ \text{ind}(r) = \text{ind}(p) - 2}} n(p, q) \cdot n(q, r) \cdot r \tag{3.17}$$

Key idea: for $p, r \in \text{Crit}(f)$ where $\text{ind}(p) - \text{ind}(r) = 2$, we can compactify

$$\hat{\mathcal{M}}(p, r)$$

the space of unparametrized index-2 flow lines, by adding in

$$\partial \hat{\mathcal{M}}(p, r) = \bigcup_{\substack{q \in \text{Crit}(f) \\ \text{ind}(q) = \text{ind}(p) - 1 = \text{ind}(r) + 1}} \hat{\mathcal{M}}(p, q) \times \hat{\mathcal{M}}(q, r).$$

Then

$$\overline{\mathcal{M}(p, r)} := \hat{\mathcal{M}}(p, r) \cup \partial \hat{\mathcal{M}}(p, r)$$

Upshot: $\overline{\mathcal{M}(p, r)}$ is a compact oriented 1-manifold, with interior $\hat{\mathcal{M}}(p, r)$.

Trivial fact: the signed count of boundary components of a 1-manifold is zero.

On the other hand, the signed count is exactly:

Lecture 13: 20 Feb

Morse homology. Today's goal: invariance. As always, M is a smooth closed manifold, $f\colon M \to \mathbb{R}$ is a Morse function, and g a Morse-Smale metric for (M, f).

Definition 12. For all $p, q \in \operatorname{Crit}(f)$,

$$\overline{\mathcal{M}}(p,q) = \bigcup \hat{\mathcal{M}}(r_1, r_2) \times \cdots \times \hat{\mathcal{M}}(r_{k-1}, r_k)$$

the union is over all $k \geq 2$ and all k-tuples $r_1, r_2, \ldots, r_k \in \operatorname{Crit}(f)$ such that $r_1 = p$ and $r_k = q$.

Here $\hat{\mathcal{M}}(r_j, r_{j+1})$ is the moduli space of unparametrized flow lines:

$$\hat{\mathcal{M}}(r_j, r_{j+1}) = \mathcal{M}(r_j, r_{j+1}) \,/\mathbb{R}\,.$$

Example (bumpy sphere).

$$\hat{\mathcal{M}}(a,d) = \hat{M}(a,d) \cup \hat{M}(a,c) \times \hat{M}(b,c) \tag{3.18}$$

and here $\hat{M}(a,c) \times \hat{M}(b,c)$ is precisely

$$\{\hat{\gamma}_1\} \times \{\hat{\gamma}_2 \times \hat{\gamma}_3\}$$

Proposition 7. $\overline{\mathcal{M}}(p,q)$ is a compact manidold of dimension

$$\operatorname{ind}(p) - \operatorname{ind}(q) - 1$$

with interior $\hat{\mathcal{M}}(p,q)$.

Elements of the boundary $\partial\overline{\mathcal{M}}(p,q)$ are (unparamterized) *broken flow lines* from p to q.

In particular, when $\#\operatorname{ind}(p) - \operatorname{ind}(q) = 2$, $\overline{\mathcal{M}}(p,q)$ is a compact 1-manifold with boundary. It was in this setting that we showed $\partial^2 = 0$ on $CM_*(M, f, g)$.

If $\operatorname{ind}(p) - \operatorname{ind}(q) > 2$, then $\overline{\mathcal{M}}(p,1)$ is stratified.

Example. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 3$, then

Philosophy: to define differentials, chain maps, chain homotopies, etc. We count elements in suitable moduli spaces. For example, the differential ∂ we had before; and to prove properties, we should the examine boundaries of moduli spaces.

To prove the Proposition requires

- compactness, and
- gluing.

compactness Given a sequence $\hat{\gamma}_1.\hat{\gamma}_2, \cdots \in \hat{\mathcal{M}}(p,q)$, there exists a convergent subsequence with limit $\hat{\gamma} \in \hat{\mathcal{M}}(p,q)$ or $\hat{\delta} * \hat{\varepsilon} \in \partial\overline{\mathcal{M}}(p,q)$ or a concatenation of more than two. Note that $*$ means to concatenate paths.

To say that

$$\hat{\gamma}_n \to \hat{\delta} * \hat{\varepsilon}$$

means there exist parametrizations $\gamma_n(t), \delta(t), \varepsilon(t)$, and sequences of real numbers $a_n, b_n \in \mathbb{R}$ such that

$$\gamma_n(t - a_n) \to \delta(t)$$

and

$$\gamma_n(t - b_n) \to \varepsilon(t)$$

where convergence is uniform on compact subsets, in other words, convergence is in C^∞_{loc}.

gluing Given

$$\hat{\delta}_1 * \cdots * \hat{\delta}_{k-1} \in \hat{\mathcal{M}}(r_1, r_2) \times \cdots \times \hat{\mathcal{M}}(r_{k-1}, r_k)$$

you can approximate it arbitrarily well by

$$\hat{\gamma} \in \hat{\mathcal{M}}(p, q)$$

where $p = r_1$, $q = r_k$.

3.5 Invariance

Problem. Given (f_0, g_0) and (f_1, g_1) choices of (Morse function, Morse-Smale metric) on M, why should

$$HM_*(M, f_0, g_0) \cong HM_*(M, f_1, g_1).$$

This is the problem of invariance of Morse homology.

To that end we will define a chain map

$$\psi\colon CM_*(M, f_0, g_0) \to CM_*(M, f_1, g_1).$$

Next time we will check the induced map of this chain map on homology is an isomorphism.

The idea is to connect (f_0, g_0) and (f_1, g_1) through a family $(f_t, g_t), 0 \leq t \leq 1$ where each

$$f_t\colon M \to \mathbb{R}$$

is a smooth function, and each g_t is a metric on M. In a generic family,

1. f_t is a Morse function for all but finitely many t,
2. g_t is Morse-Smale for (M, f_t) for the same all but finitely many t,
3. moreover, the following construction has the stated properties:

pic

Want to define, for each $p_0 \in \mathrm{Crit}(f_0)$,

$$\psi(p_0) = \sum_{q_1 \in \mathrm{Crit}(f_1)} n(p_0, q_1) \cdot q_1$$

for some suitable coefficients $n(p_0, q_0)$.

Form a kind of gradient-like vector field for $\{(f_t, g_t)\}$ on $M \times I$ and define $n(p_0, q_1)$ as a signed count of flow lines.

Without checking details, define

$$V \in \mathrm{Vect}(M \times [0,1])$$

to interpolate from $\nabla_{g_0} f_0$ to $\nabla_{g_1} f_1$ as follows: pick a smooth function

$$\beta\colon [0,1] \to \mathbb{R}_{\geq 0}$$

such that $\beta(0) = 0$, $\beta(1) = 0$, and $\beta(t) > 0$ for $0 < t < 1$

(component of V in the $\frac{\partial}{\partial t}$ direction). Now for $x \in M$, $t \in [0,1]$,

$$V(x,t) = \nabla_{g_t} f_t(x) + \beta(t) \cdot \frac{\partial}{\partial t}.$$

$$\mathcal{M}(p_0, q_1) = \{\gamma\colon \mathbb{R} \to M \times [0,1] \colon \}$$

Proposition 8. For generic $\{(f_t, g_t)\}$,

Lecture 15: 22 Feb

Goal: define a quasi-isomorphism (chain map that induces isomorphism on homology)

$$\psi\colon CM_*(M, f_0, g_0) \to CM_*(M_1, f_1, g_1).$$

Interpolate by $(f_t, g_t), 0 \leq t \leq 1$ of (smooth function,metric)'s. We deill define a gradient-like vector field on $M \times [0,1]$ by

$$V(x,t) := \beta(t)\frac{\partial}{\partial t} + \nabla_{g_t} f_t$$

where

$$\beta\colon [0,1] \to \mathbb{R}_{\mathbb{R}\geq 0}$$

such that $\beta(t) = 0$ if and only if $t = 0, 1$. The zeroes, i.e. critical points of V are $\mathrm{Crit}(f_0) \times 0$ and $\mathrm{Crit}(f_1) \times 1$. Picture:

pic

Critical points have indices just like before, equal

$$\dim W_u(p, -V)$$

where p is a critical point of V. If $p_0 \in \mathrm{Crit}(f_0)$ and $q_1 \in \mathrm{Crit}(f_1)$, then $(p_0, 0)$ has index $\mathrm{ind}(p_0) + 1$, and $(q_1, 1)$ has index $\mathrm{ind}(q_1)$.

Let ϕ_t denote the time-t flow of $-V$. Suppose

$$\gamma \colon \mathbb{R} \to M \times [0,1]$$

is a flow line of $-V$, i.e.:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -V \circ \gamma.$$

We have

$$\lim_{t \to \pm\infty} \gamma(t)$$

are critical points of $-V$, just as before (exercise). Hence either

- $\mathrm{Im}(\gamma) \subset M \times 0$ and γ is a flow line of $-\nabla_{g_0} f_0$
- $\mathrm{Im}(\gamma) \subset M \times 1$ and γ is a flow line of $-\nabla_{g_1} f_1$
- $\lim_{t \to -\infty} \gamma(t) =$

We can now define for $p_0 \in \mathrm{Crit}(f_0), q_1 \in \mathrm{Crit}(f_1)$,

$$\mathcal{M}(p_0, q_1) \coloneqq \{\}$$

Proposition 9. $\mathcal{M}(p_0, q_1)$ is a smooth manifold for all p_0, q_1, for generic $\{(f_t, g_t)\}$; of dimension

$$\dim \mathcal{M}(p_0, q_1) = \mathrm{ind}(p_0) + 1 - \mathrm{ind}(q_1).$$

Definition 13.

$$\hat{\mathcal{M}}(p_0, q_1) \coloneqq \mathcal{M}(p_0, q_1)/\mathbb{R}$$

(mod out reparametrization). Then

$$\dim \hat{\mathcal{M}}(p_0, q_1) = \mathrm{ind}(p_0) - \mathrm{ind}(q_1).$$

Define

$$\psi \colon CM_*(f_0, g_0) \to CM_*(f_1, g_1)$$

on generators by

$$\psi(p_0) = \sum_{q_1 \in \mathrm{Crit}(f_1), \mathrm{ind}(q_1) = \mathrm{ind}(p_0)} \#\hat{\mathcal{M}}(p_0, q_1) \cdot q_1.$$

To prove firstly that ψ is a chain map, we compactify the moduli spaces and study their boundaries. For $p_0 \in \mathrm{Crit}(f_0)$, and $q_1 \in \mathrm{Crit}(f_1)$, we compactify $\hat{\mathcal{M}}(p_0, q_1)$ by adding in all unparametrized broken flow lines of $-V$ from $(p_0, 0)$ to $(q_1, 1)$:

pic

$$\overline{\mathcal{M}}(p_0, q_1) =$$

Proposition 10. $\overline{\mathcal{M}}(p_0, q_1)$ is compact with sated interior and bounary.

Corollary. If $\text{ind}(p_0) = \text{ind}(q_1) + 1$, then $\overline{\mathcal{M}}(p_0, q_1)$ is a compact 1-manifold with boundary all broken flow lines of the form

$$\hat{\mathcal{M}}(p_0, r_0) \times \hat{\mathcal{M}}(r_0, q_1)$$

where

$$\text{ind}(p_0) - \text{ind}(r_0) = 1.$$

$$\text{ind}(r_0) = \text{ind}(q_1)$$

Lecture 16: 24 Feb

Suppose $(f_0, g_0), (f_1, g_1)$ are two choices (Morse function, Morse-Smale metric) on M (closed and smooth). Suppose (f_t, g_t) and (f_t', g_t') are two different interpolations between (f_0, g_0) and (f_1, g_1). Then we described: there exist affiliated chain maps

$$\psi, \psi' \colon CM_*(M, f_0, g_0) \to CM_*(M, f_1, g_1).$$

The goal: show ψ and ψ' are chain homotopic, i.e. there exist

$$H \colon CM_*(M, f_0, g_0) \to CM_{**}(M, f_1, g_1)$$

such that

$$H\partial_0 + \partial_1 H + \psi + \psi' = 0$$

so (mod 2, if not careful about orientations), ψ and ψ' are chain homotopic.

We can interpolate the interpolations: there exists a family (f_t^s, g_t^s), $0 \leq t \leq 1, 0 \leq s \leq 1$, such that

- $(f_0^s, g_0^s = (f_0, g_0)$
- $(f_1^s, g_1^s) = (f_1, g_1)$
- $(f_t^0, g_t^0 = (f_t, g_t)$
- $f_t', g_t') = (f_t', g_t')$

Picture:

The critical points $\text{Crit}(-V)$, i.e.

$$\text{Crit}(-V) = \{(x, s, t) \colon M \times B \colon -V(x, s, t) = 0\}$$

must satisfy

$$\text{Crit}(f_0) \times N \cup \text{Crit}(f_1) \times S.$$

Then

$$\text{ind}_{-V}(p_0, N) = \text{ind}_f(p_0) + 2$$

$$\text{ind}(q_1, S) = \text{ind}(q_1).$$

Proposition 11. For $p_0 \in \mathrm{Crit}(f_0)$ and $q_1 \in \mathrm{Crit}(f_1)$, let

$$\mathcal{M}(p_0, q_1) = \left\{ x \in M \times B\colon \lim_{t\to-\infty} \phi_t^{-V}(x) = (p_0, N), \lim_{t\to+\infty} \phi_t^{-V}(x) = (q_1, S) \right\}.$$

And define

$$\hat{\mathcal{M}}(p_0, q_1) \coloneqq \mathcal{M}/\mathbb{R}.$$

Then for generic (f_t^s, g_t^s) interpolating generic (f_t, g_t) and (f_t', g_t'), $\hat{\mathcal{M}}(p_0, q_1)$ is a smooth manifold of dimension

$$\mathrm{ind}(p_0) - \mathrm{ind}(q_1) + 1$$

Define

$$H\colon CM_*(f_0, g_0) \to CM_{*+1}(f_1, g_1)$$

on generators by

$$H(p_0) = \sum_{\substack{q\in\mathrm{Crit}(f_1)\\ \mathrm{ind}(q_1)=\mathrm{ind}(p_0)+1)}} \#\hat{\mathcal{M}}(p_0, q_1) \cdot q_1.$$

Proposition 12. There exists a compactification $\overline{\mathcal{M}}(p_0, q_1)$ by "broken" flow linesof $-V$ from p_0 to q_1.

If $\dim \overline{\mathcal{M}}(p_0, q_1) = 1$ then its boundary consists of

1. $\hat{\mathcal{M}}(p_0, r_0) \times \hat{\mathcal{M}}(r_0, q_1)$ for all $r_0 \in \mathrm{Crit}(f_0)$ such that

   $$\dim \hat{\mathcal{M}}(p_0, r_0) = \dim \hat{\mathcal{M}}(r_0, q_1)$$

   i.e.

   $$\mathrm{ind}(r_0) = \mathrm{ind}(p_0) - 1 = \mathrm{ind}(q_1) + 1$$

   Hence

   $$\sum_{\substack{r_0\in\mathrm{Crit}(f_0)\\ \mathrm{ind}(r_0)=\mathrm{ind}(p_0)-1}} \#(\hat{\mathcal{M}}(p_0, r_0) \times \hat{\mathcal{M}}(r_1, q_1))$$

   equals the coefficients of q_1 in $H \circ \partial_0(p_0)$.

2. $$\bigcup_{r_1} \hat{\mathcal{M}}(p_0, r_1) \times \hat{\mathcal{M}}(r_1, q_1)$$

   where the union is over all r_1 $\mathrm{Crit}(f_1)$ of

   $$\mathrm{ind}(r_1) = \mathrm{ind}(p_0) - 1 = \mathrm{ind}(q_1) + 1.$$

   The number of this is the coefficient of q_1 in $\partial \circ H(p_0)$

3. Breaking on front. The coefficient of q_1 in $\psi(p_0)$ counts # of points in this piece of boundary.

4. Breaking on back. The coefficient of q_1 in $\psi'(p_0)$ counts # of points in this piece of boundary.

For all $(f_0, g_0), (f_1, g_1)$, there exists a preferred isomorphism

$$\psi_*^{01} \colon HM_*(M, f_0, g_0) \to HM_*(M, f_1, g_1)$$

such that for all (f_2, g_2),

$$\psi_*^{02} = \psi_*^{12} \circ \psi_*^{01}$$

Define

$$HM_*(M) \subset \prod_{\substack{(f,g) \text{ is} \\ \text{(Morse function, Morse-Smale metirc)}}} HM_*(M, f, g)$$

Lecture 17: 27 Feb

3.6 The Morse Product

Let M be a smooth closed manifold. Recall, cup product on ordinary cohomology:

$$\smile \colon H^a(M) \otimes H^b(M) \to H^{a+b}(M)$$

Interpretation: if $\alpha \in H^a(M)$ and $\beta \in H^b(M)$, then $PD[\alpha] \in H_i(M)$, and $PD[\beta] \in H_j(M)$, where $i = n - a$, $j = n - b$. Imagine representing these homology classes by submanifolds A and B, respectively. Position A and B transversally to one another to get $C = A \cap B$ a smooth manifold of dimension $i + j - n$:

$$\operatorname{codim}(C, A) = \operatorname{codim}(B, M)$$
$$\dim A - \dim C = \dim A - \dim B$$
$$vdots$$

Now

$$PD[\alpha \smile \beta] = [C]$$

thus

$$\alpha \smile \beta \in H^{2n-i-j}(M) = H^{a+b}(M).$$

Hence by taking Poincare duals, we can view cup product as a product on homology, i.e. the *intersection product*:

$$H_i(M) \otimes H_j(M) \to H_k(M)$$

where $i + j - k = n$. The unit for the intersection product is the fundamental class $[M] \in H_n(M)$,

Suppose (f, g) is chosen for M as in Morse homology. Which cycle (in Morse homology) represents $[M]$ (in ordinary homology)?

$$[M] \text{ is represented by } \sum_{p \in \mathrm{Crit}_n(M)} p$$

Notation. $\mathrm{Spec}(f) \coloneqq \{f(p) \colon p \in \mathrm{Crit}(f)\})$

Why is it a cycle?

$$\partial \sum_{p \in \mathrm{Crit}_n(M)} p = \sum_{q \in \mathrm{Crit}_{n-1}(f)} n_q \cdot q$$

Want to show: $n + q = 0$. $W_s(q)$ is 1-dimensional, thus there exist two index-1 flow lines into q from the maxima of f.

To define the cup product in Morse homology Define a chain level product

$$m\colon CM_i \otimes CM_j \to CM_k$$

by choosing a generic triple $(f_0, g_0), (f_1, g_1), (f_2, g_2)$, and defining

$$m\colon CM_1(f_0, g_0) \otimes CM_j(f_1, g_1) \to CM_k(f_2, g_2).$$

Given $p_0 \in \mathrm{Crit}(f_0), p_1 \in \mathrm{Crit}(f_1)$, let

$$m(p_0, p_1) = \sum_{\substack{p_2 \in Crit(f_2) \\ \mathrm{ind}(p_0 + \mathrm{ind}(p_1) = n + \mathrm{ind}(p_2)}} n(p_0, p_1, p_2) \cdot p_2$$

where

$$n(p_0, p_1, p_2) = \#\mathcal{M}(p_0, p_1, p_2).$$

For $p_i \in \mathrm{Crit}(f_i), i = 0, 1, 2,$

$$\mathcal{M}(p_0, p_1, p_2) = pic$$

Proposition 13. For a generic $(f_i, g_i), i = 0, 1, 2$, $\mathcal{M}(p_0, p_2, p_2)$ is a manifold of dimension

$$\mathrm{ind}(p_0) + \mathrm{ind}(p_1) - \mathrm{ind}(p_2) - n.$$

So if $\mathrm{ind}(p_0) + \mathrm{ind}(p_1) - \mathrm{ind}(p_2) - n = 0$, then $\mathcal{M}(p_0, p_1, p_2)$ is a 0-manifold and

$$\#\mathcal{M}(p_0, p_1, p_2) \in \mathbb{Z}.$$

Remark. There is no $\mathbb{R}$-action to divide out.

We want to show m obeys the Leibniz rule in order to get an induced product on homology; WTS:

$$\partial_2 \circ m(p_0, p_1) = m(\partial_0 p_0, p_1) + m(p_0, \partial_1 p_1)$$

Proposition 14. For all p_0, p_1, p_2, there exists a compactification of $\mathcal{M}(p_0, p_1, p_2)$ by "broken Y's":

If $\dim \overline{\mathcal{M}}(p_0, p_1, p_2) = 1$, then $\partial \overline{\mathcal{M}}(p_0, p_1, p_2)$ consists of broken Ys of the forms

Chapter 4

Hamiltonian Floer Homology

Lecture 18: 1 Mar

Towards Hamiltonian Floer homology. A few structures on $\mathbb{C}$:

$$v = x_1 + iy_1, w = x_2 + iy_2 \in \mathbb{C}.$$

There is an endomorphism $J\colon \mathbb{C} \to \mathbb{C}$:

$$J(v) = i \cdot v.$$

The key property is that

$$J^2 = -\operatorname{Id}.$$

More generally, if $E \to M$ is a vector bundle over a smooth manifold, and $J \in \operatorname{End}(E)$ i.e.

$$J\colon E_p \to E_p \quad \forall p \in M$$

varying smoothly with p. If this J satisfies $J^2 = -\operatorname{Id}$, then we call J an *almost complex structure*. So in the previous setting, J is an almost complex structure on the tangent bundle.

There is a Hermitian product

$$H(v, w) = v \cdot \overline{w}.$$

If we write out in coordinates,

$$\begin{aligned} H(v, w) &= v \cdot \overline{w} && (4.1)\\ &= (x_1 + iy)(x_2 - iy_2) && (4.2)\\ &= (x_1x_2 + y_1y_2) + i(x_1y_2 - y_1x_2) && (4.3)\\ &= g(\overrightarrow{v}, \overrightarrow{w}) + i\omega(\overrightarrow{v}, \overrightarrow{w}) && (4.4) \end{aligned}$$

where g is the standard metric on $\mathbb{R}^2$ and ω is the standard symplectic form on $\mathbb{R}^2$:

$$\omega(\overrightarrow{v}, \overrightarrow{w}) = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Fundamental property:

$$\omega(v, J \cdot w) = g(v, w).$$

Definition 14. A *Kahler manifold* is a complex manifold with affiliated almost complex structure J, symplextic form $\omega, and g$, satisfying

$$\omega(\cdot, J\cdot) = g(\cdot,\cdot).$$

Example. $\mathbb{C}^n$, $\mathbb{C}P^n$, and complex algebraic varieties

Given (M,ω) a symplextic manifold, and J an a.c.c. on M such that the relation holds, i.e. $\omega(\cdot, J\cdot)$ is a metric, we say J is compatible with ω.

Goal: estimate $\#\operatorname{Fix}(\phi)$ where $\phi \in \operatorname{Ham}(M,\omega)$. Recall: ϕ is the time-1 flow of a time-dependent Hamiltonian

$$H = H_t \in C^\infty(M \times [0,1]).$$

We will request that H_t is 1-periodic. So

$$H = H_t \in C^\infty\left(M \times {}^{\mathbb{R}}/_{\mathbb{Z}}\right).$$

Recall also that $\operatorname{Fix}(\phi)$ is in one-to-one correspondence with closed orbits of $\phi^t_{H_t}, 0 \le t \le 1$.

We had a way to characterize this: there is the action functional

$$\mathcal{A} = \mathcal{A}_{H_t} = \mathcal{A}_H \colon \mathcal{L}_M \to \mathbb{R}$$

where

$$\mathcal{L}(M) = \{\text{contractible loops } \gamma\colon \mathbb{R}/\mathbb{Z} \to M\}.$$

The action functional is defined by

$$\mathcal{A}(\gamma) = \int_0^1 H_t \circ \gamma(t)\mathrm{d}t + \underbrace{\int_{D^2} \hat{\gamma}^*\omega}_{\text{symplectic area}}$$

where $\hat{\gamma}$ is the "capping": $\hat{\gamma}\colon D^2 \to M$ such that $\hat{\gamma}\big|_{\partial D^2} = \gamma$.

Exercise: relate this to the earlier action we wrote down for an exact (M,ω).

To guarantee that $\mathcal{A}$ is independent of the choice of capping $\hat{\gamma}$, we could assume $\pi_2(M) = 0$, or $[\omega]\cdot\pi_2(M) = 0$. This is the condition of M being *symplectically aspherical*.

Proposition 15.

$$\operatorname{Crit}(\mathcal{A}_{H_t}) = \{\text{contractible closed orbits of } \phi^t_{H_t}, 0 \le t \le 1\}$$

Goal (Arnold Conjecture plus some conditions): $\#\operatorname{Crit}(\mathcal{A}_{H_t}) \ge \dim H_*(M;\mathbb{F}_2)$.

We'd like to interpret $\mathcal{A}$ as a Morse function on $\mathcal{L}(M)$. Then do Morse homology with it to conclude

$$\#\operatorname{Crit}(\mathcal{A}) \ge \dim H_*(\mathcal{L}(M),\mathbb{F}_2) = \dim H_*(M,\mathbb{F}_2).$$

Exercise: Prove $\mathcal{L}(M)$ is homotopy equivalent to M.

We need a metric on $\mathcal{L}(M)$ and study gradient flow of $\mathcal{A}$ with respect to the metric. What is a suitable metric? The metric is a positive definite inner product on the tangent spaces

$$T_\gamma \mathcal{L}(M)$$

which has to vary smoothly in γ. (Recall) Given $X, Y \in T_\gamma \mathcal{L}(M)$ we want to define

$$\langle X, Y \rangle.$$

We write $X = X_t$, and $Y = Y_t$ where $t \in \mathbb{R}/\mathbb{Z}$, such that $X_t, Y_t \in T_{\gamma(t)} M$.

pic

i.e. X and Y are vector fields on

$$\gamma^* TM$$

Suppose we choose a metric g on M. Define $\langle \cdot, \cdot \rangle$ on the loop space $\mathcal{L}(M)$ by integrating:

$$\langle X, Y \rangle \coloneqq \int_0^1 g(X_t, Y_t) \mathrm{d}t.$$

> **Remark.** If we had a time dependent family of metrics g_t we could also define this (just adding the additional t's for the gs).

We need to define the gradient of $\mathcal{A}$ with respect to this metric: $\nabla_{\langle \cdot, \cdot \rangle} \mathcal{A}$, and then study for

$$u \colon \mathbb{R}_s \to \mathcal{L}(M),$$

$$\frac{\mathrm{d}u}{\mathrm{d}s} - \nabla_{\langle \cdot, \cdot \rangle} \mathcal{A} \circ u(s).$$

gradient flow???

Lecture 19: 3 Mar

Edit of last time: we wete considering $H(v, w) = v \cdot \overline{w}$,

$$H(v, Jw) = -i \cdot H(v, w)$$

writing out in real and imaginary parts, the LHS is

$$g(v, Jw) + i \cdot \omega(v, Jw)$$

and the RHS is

$$\omega(v, w) - i \cdot g(v, w)$$

????

Set-up: (M, ω) is closed symplectic manifold, and

$$H = H_t \in C^\infty \left(M \times {}^{\mathbb{R}} /_{\mathbb{Z}} \right)$$

and suppose $\mathrm{Fix}(\phi_H^1)$ consists of non-degenerate fixed points. Goal is to prove

$$\# \mathrm{Fix}(\phi_H^1) \geq \dim H_*(M)$$

assuming e.g. $\pi_2(M) = 0$.

We observe $\mathrm{Fix}(\phi_H^1)$ are in one-to-one correspondence with $\mathrm{Crit}(\mathcal{A}_H)$. And now we want to do Morse homology with $\mathcal{A}_H$ on $\mathcal{L}(M)$. This involves putting a metric $\langle\cdot,\cdot\rangle$ on $\mathcal{A}_H$ and studying (negative) gradient flow. We defined the metric last time, i.e. first by picking a g on M, and then....

We will only consider g is of the form

$$g(v, w) = \omega(v, Jw)$$

where J is an almost complex structure on M. So

$$\langle X, Y\rangle = \int_0^1 \omega(X_t, Jy_t)\mathrm{d}t.$$

So the gradient will be

$$\nabla_{\langle\cdot,\cdot\rangle}\mathcal{A}_{H_t} = \nabla\mathcal{A}$$

is defined implicitly by

$$\mathrm{d}\mathcal{A} = \langle\nabla\mathcal{A}, -\rangle.$$

Recall:

$$(\mathrm{d}\mathcal{A})_\gamma(Y) = \int_0^1 \omega\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - X_{H_t}\circ\gamma(t), Y_t\right)\mathrm{d}t$$

recovers the correspondence

$$\mathrm{Crit}(\mathcal{A}) \leftrightarrow \text{ closed Hamiltonian orbits.}$$

Going back to what we want,

$$\langle\nabla\mathcal{A}, Y\rangle = \int_0^1 \omega(\nabla\mathcal{A})_\gamma(t), J\cdot Y_t)\mathrm{d}t$$

where $(\nabla\mathcal{A})_\gamma(t)$ is the tangent vector to M at $\gamma(t)$. This is equal to

$$= \int_0^1 (-J(\nabla\mathcal{A})_\gamma(t), Y_t) \tag{4.5}$$

using the symmetry and metric and the skew-symmetry of ω.

Now because ω is non-degenerate and Y_t is arbitrary,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - X_{H_t}\circ\gamma(t) = -J\cdot(\nabla\mathcal{A})_\gamma(t)$$

hence

$$(\nabla\mathcal{A})_\gamma(t) = J\cdot\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - X_{H_t}\circ\gamma(t)\right)$$

What is the (negative) gradient flow equation? Recall, in the finite dimensional case, it is $\gamma\colon\mathbb{R}\to M$ satisfying

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = -\nabla_g f\circ\gamma(t).$$

Translated to our setting, we obtain

$$u\colon\mathbb{R}_s\to\mathcal{M}$$

such that

$$\frac{\mathrm{d}u}{\mathrm{d}s}(s) = -\nabla\mathcal{A} \circ u(s)$$

picture is a cylinder.

i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}s}(s,t) = -\nabla\mathcal{A} \circ u(s,t) = -J\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - X_{H_t} \circ \gamma(t)\right)$$

where $\gamma 9t) = u(s,t)$.

Rearranged, with $\gamma = u(s,\cdot)$, we get

$$\frac{\partial u}{\partial s}(s,t) + J \cdot \left(\frac{\partial u}{\partial t}(s,t) - X_{H_t} \circ u(s,t)\right) = 0$$

abbreviated as

$$\partial_s u + j \cdot (\partial_t u - X_{H_t} u) = 0$$

holds for all $(s,t) \in \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$.

Note is H_t is constant, then this reduces to

$$\partial_s u + J \cdot \partial_t u = 0.$$

This is the Cauchy-Riemann Equation for a cylinder

$$u \colon \mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to (M, J).$$

To justify the name,

$$u \colon \mathbb{C} \supset \Omega \to \mathbb{C}$$

is holomorphic if and only if

$$\partial_s u + J \cdot \partial_t u = 0$$

where J denotes the standard almost complex structure on $\mathbb{C}$ (i.e. multiplication by i) using complex coordinate $z = s + it$.

$$\partial_s u + j \cdot (\partial_t u - X_{H_t} u) = 0$$

is often called teh Cauchy-Riemann Floer Equation (CRF).

There is a trick due to Gromov to convert CRF into a Cauchy-Riemann equation using a time-dependent almost complex structure:

$$\partial_s \tilde{u} + \tilde{J}_t \partial_t \tilde{u} = 0.$$

Where we are headed: we will build a chain complex freely generated by

$$\mathrm{Crit}(\mathcal{A}) \leftrightarrow \text{ closed Hamiltonian orbits}$$

and whose differential counts certain so-called pseudoholomorphic cylinders:

$$\partial_\gamma = \sum_{\gamma'} n(\gamma, \gamma') \cdot \gamma'.$$

Lecture 20: 13 Mar

Setting: (M, ω) closed symplectic manifold, and $H = H_t \in C^\infty(M \times \times \mathbb{R}/\mathbb{Z})$ a time-dependent Hamiltonian which is 1-periodic. We are interested in

$$Orb(H) = \{\gamma \mathbb{R}/\mathbb{Z} \to M \mid \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_{H_t} \circ \gamma(t)\}$$

the 1-periodic orbits, or loops in M that satisfy Hamilton's equation (we also impose contractibility of γ). We showed, under assumption of symplectic fillable, this set is equal to $\mathrm{Crit}(\mathcal{A}_H)$, which is $\subset \mathrm{Fix}(\phi^1_{H_t})$ (? make sure).

Assume that all $\gamma \in Orb(H)$ are non-degenerate. Recall, that this means the graph of $\phi^1_{H_t}$,

$$\Gamma(\phi^1_{H_t}) = \{(x, \phi^1_{H_t}(x)) \mid x \in M\} \subset M \times M$$

is transversal to the diagonal $\{(x, x) \mid x \in M\}$.

Idea. Do Morse theory with

$$\mathcal{A}_H \colon \mathcal{L}(M) \to \mathbb{R}.$$

To do so, we picked an almost-complex tructure J on M which is compatible with ω in the sense that

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

defines a metric on M.

We then get a metric $\langle \cdot, \cdot \rangle$ on $\mathcal{L}(M)$, and a gradient $\nabla \mathcal{A}_H$ defined with the metric and the negative flow equation:

$$u \colon \mathbb{R}_s \to \mathcal{L}(M)$$

smooth, such that

$$\frac{\mathrm{d}u}{\mathrm{d}s}(s) = -\nabla \mathcal{A} \circ \gamma(s).$$

Unpacking this gives

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{H_t} \circ u\right) = 0$$

the Cauchy-Riemann-Floer equation, a.k.a. the perturbed Cauchy-Riemann equation.

Exercise: See that

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t}\right) = 0$$

for

$$u \colon \mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to \mathbb{C}$$

and J being multiplication by i, is exactly the Cauchy-Riemann equaitons; that is, u obeys the Cauchy-Riemann equations if and only if u is holomorphic.

Problem. Some issues with doing Morse theory in this setting:

1. By contrast with the negatie gradient equation on a finite-dimensional manifold, it is not in general possible to solve the negative gradient flow equation for given initial data $u(0) = \gamma \in \mathcal{L}(M)$ (analog in finite-dimensions is that given any point on the manifold, by the existence and uniquness of oslutions to ODEs there always a path? with this initial condition so each point sits in some unstable manifold). This obsticle was noted prior to Floer by Moser. Thus, there is no globally defined flow ϕ_s, $s \in \mathbb{R}$ on $\mathcal{L}(M)$. Because CRF is in unsolvable in general, we cannot decompose $\mathcal{L}(M)$ into unstable manifolds such as
$$W(\gamma_0) = \{\gamma \in \mathcal{L}(M) \mid \lim_{s\to-\infty} \phi_s(\gamma) = \gamma_0\}$$
for $\gamma_0 \in \mathrm{Crit}(\mathcal{A}_H)$.
2. These unstable manifolds $W(\gamma_0)$ are typically infinite-dimensional. This causes difficulties in defining index $\mathrm{ind}(\gamma_0)$. Similarly, the stable manifolds are typically infinite-dimensional, making it doubtful that $W_u(\gamma_0)\cap W_s(\gamma_1)$ is finite-dimensional.

   Given $u(\cdot)\colon \mathbb{R} \to \mathcal{L}(M)$ satisfying CRF, then for any $s \in \mathbb{R}$, then
$$u(s+\cdot)\colon \mathbb{R} \to \mathcal{L}(M)$$
   also obeys CRF.

   A better way to define $W(\gamma_0)$ is
$$W(\gamma_0) \coloneqq \left\{u \in \mathcal{L}(M) \mid u \text{ satisfies CRF; } \lim_{s\to\infty} u(s)\gamma_0\right\}.$$
   S the flow ϕ_s is defined on $W(\gamma_0)$ by
$$\phi_s(u(\cdot)) = u(s+\cdot).$$
3. Given $u\colon \mathbb{R} \to \mathcal{L}(M)$ obeying CRF,
$$\lim_{s\to\pm\infty} u(s)$$
   may not converge.

Floer overcame those issues by mimicking Morse homology. For a given pair $\gamma_0, \gamma_1 \in Orb(H_t)$, define the moduli space
$$\mathcal{M}(\gamma_0,\gamma_1) \coloneqq \{u\colon \mathbb{R} \to \mathcal{L}(M) \mid u \text{ satisfies CRF; } \lim_{s\to-\infty} u(s) = \gamma_0, \lim_{s\to\infty} u(s) = \gamma_1\}.$$
This admits an $\mathbb{R}$-action:
$$s \cdot u(\cdot) = u(s+\cdot).$$
So we can define the divided/unparametrized moduli space
$$\hat{\mathcal{M}}(\gamma_0,\gamma_1) = \mathcal{M}(\gamma_0,\gamma_1)/\mathbb{R}.$$

We alos need an analog of the Morse-Smale condition on our choice of almost-complex structure, in order to ensure these (divided moduli spaces) are finite-dimensional manifolds. TO accomplish that, we need to allow J to have a t-dependence $J = J_t$. Then all the $\hat{\mathcal{M}}(\gamma_0,\gamma_1)$ are finite-dimensional manifolds.

Now define

$$CF(M, \omega, H, J)$$

to be the $\mathbb{F}_2$ vector space finitely generated by $\text{Crit}(\mathcal{A}_H)$.

Define $\partial CF \to CF$ by, for $\gamma_0 \in \text{Crit}(\mathcal{A}_H)$,

$$\partial(\gamma_0) = \sum_{\gamma_1 \in \text{Crit}(\mathcal{A}_H) : \ \dim \hat{\mathcal{M}}(\gamma_0, \gamma_1)} \#\hat{\mathcal{M}}(\gamma_0, \gamma_1) \cdot \gamma_1.$$

Theorem 8 (Floer)**.** Suppose that

- (M, ω) closed symplctic manifold.
- $\pi_2(M) = 0$.
- $H \in C^\infty(M \times \mathbb{R}/\mathbb{Z})$
- all closed orbits are non-degenerate
- $J = J_t$ is a generic times-dependent a.c.s. on M compatible with ω (analog of Morse-Smale condition)

Then

- $\partial^2 = 0$, so we get a chain complex
- The chain homotopy type is independent of (H, J)
- there exists a specific time-independet (H, J) ssuch that
  - H is a Morse function on M
  - $g9\cdot, \cdot) = \omega(\cdot, J\cdot)$ is Morse-Smale with respect H
  - $CF(M, \omega, J, H) \cong CM(M, H, g)$

Consequently,

$$\#Orb(H) = \$\,\text{Crit}(\mathcal{A}_H) \geq \dim HM_*(M; \mathbb{F}_2) = \dim H_*(M; \mathbb{F}_2)$$

Lecture 21: 15 Mar

Recap:

- (M, ω) a closed symplectic manifold
- $\pi_2(M) = 0$
- $H = H_t \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z})$
- $\text{Orb}(H)$ are the closed contractible orbits, and we assume all are non-degenerate
- $J = J_t$ smooth, time-dependent almost-complex structure on M compatible with M

Cauchy-Riemann-Floer Equation:

$$u\colon \mathbb{R}_s \to \mathcal{L}(M)$$

$$\partial_s u + J_t(\partial_t U - X_{H_t} \circ u) = 0.$$

For a pair of orbits $\gamma_0, \gamma_1 \in \mathrm{Orb}(H)$, we defined $\mathcal{M}(\gamma_0, \gamma_1)$, which has an $\mathbb{R}$-action:

$$s \cdot u(\cdot) = u(s + \cdot).$$

Then we defined $\hat{M}$.

Theorem 9. For generic J_t (analog of Morse-Smale condition), $\hat{M}(\gamma_0, \gamma_1)$ is a smooth manifold for all $\gamma_0, \gamma_1 \in \mathrm{Orb}(H)$. Moreover,

$$\dim \hat{M}(\gamma_0, \gamma_1) = \mu(\gamma_0) - \mu(\gamma_1) - 1$$

for a suitable Maslov index μ (to be defined later).

Given such a choice $J = J_t$, define

$$CF(M, \omega, H, J) = \mathbb{F}_2 \cdot \mathrm{Orb}(H)$$

and define the boundary operator

$$\partial\colon CF(M, \omega, H, J) \to CF(M, \omega, H, J)$$

bia

$$\partial(\gamma_0) = \sum_{\substack{\gamma_1 \in \mathrm{Orb}(H) \\ \mu(\gamma_1) = \mu(\gamma_0) - 1}} \#\hat{\mathcal{M}}(\gamma_0, \gamma_1) \cdot \gamma_1.$$

Theorem 10.

1. $\partial^2 = 0$ so $CF_*(M, \omega, H, J)$ is a chain complex.
2. The chain homotopy type is independent of (H, J).
3. There exist time-independent (H, J) such that

$$CF_*(M, \omega, H, J) \approx CF_{*+n}(M, H, g)$$

$(M = M^{2n})$ where $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Hence

$$\#\,\mathrm{Orb}(H) \geq \dim H_*(M; \mathbb{F}_2).$$

4.1 Almost Complex Structures

Let M be a smooth manifold. An *almost complex strcuture* on M is an element $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$. If (M, ω) is a symplectic manifold, we say an almost complex structure J on (M, ω) is *compatible with* ω if

$$g(v, w) = \omega(v, Jw)$$

defines a metric, i.e.

- $\omega(v, Jv) \geq 0$ with equality if and only if $v = 0$
- $\omega(v, Jv)$ is symmteric, which is equivalent to: J preserves ω i.e.

$$\omega(v, w) = \omega(Jv, Jw).$$

Definition 15. Denote by $J(M, \omega)$ the space of all smooth almost complex structures on M which are compatible with ω.

Theorem 11. For any (M, ω), the space $J(M, \omega)$ is non-empty and contractible.

The contractibility condition is useful to choose some J as acuxilliary data and used for prooving maps between different choices.

Proof. We first work fiberwise. Fix any $x \in M$ and consider

$$\{J_x \colon T_xM \to T_xM\}$$

where

- $J_x^2 = \mathrm{Id}$
- $\omega_x(v_x, w_x) = \omega(J_xv_x, J_xw_x) \forall v_x, w_x \in T_xM$
- $\omega(v_x, Jv_x) \geq 0$ and equality iff. $v_x = 0$.

WTS: prove this $J_x(M, \omega)$ is non-empty and contractible. This is the crux of the argument, it will be followed by standard differential topology.

By linear Darboux, we can identify (symplectomorphism)

$$(T_xM, \omega) \cong (\mathbb{R}^{2n}, \omega_0).$$

So now it suffices to show $J_0(\mathbb{R}^{2n}, \omega_0)$ is non-empty and contractible.

There exists a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ for $\mathbb{R}^{2n}$ with dual basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}y_1, \ldots, \mathrm{d}y_n$ and

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

In this basis, given vectors $\vec{v}, \vec{w} \in \mathbb{R}^{2n}$,

$$\omega_0(v, w) = v^T \begin{pmatrix} 0_{n\times n} & \mathrm{Id}_{n\times n} \\ -\mathrm{Id}_{n\times n} & 0_{n\times n} \end{pmatrix} w = J_0$$

Note, $J_0^2 = -\mathrm{Id}$. It follows that $-J_0 \in J_0(\mathbb{R}^{2n}, \omega_0)$. Check

-

- $$\begin{aligned}\omega_0(-J_0v, -J_0w) &= v^T J_0^T J_0 J_0 w && (4.6)\\ &= v^T(-J_0^3)w && (4.7)\\ &= v^T J_0 w && (4.8)\\ &= \omega_0(v, w) && (4.9)\end{aligned}$$

- $\omega(v, -J_0, v) = v^T J_0(-J_0)v = v^T v$.

Pick $J \in J_0(\mathbb{R}^{2n}, \omega_0)$, i.e.

-

Consider

$$\begin{aligned}(J_0J)^T &= J^T J_0^T && (4.10)\\ &= -J^T J_0 && (4.11)\\ &\overset{a.c.s.}{=} J^T J_0 J^2 && (4.12)\\ &\overset{symmetry}{=} J_0 J && (4.13)\end{aligned}$$

this shows J_0J is a symmetric positive definite matrix, and it is in $\mathrm{Sp}(2n)$.

□

Lecture 22: 17 Mar

Setting: (M, ω) symplectic manifold, denote by $J(M, \omega)$ the set of smooth almost complex structures on M that are compatible with ω. Main Theorem: $J(M, \omega)$ is non-empty and contractible.

Steps of proof:

1. We show that
$$J_0(\mathbb{R}^{2n}, \omega_0)$$
is non-empty and contractible. Recall, for $\vec{v}, \vec{w} \in \mathbb{R}^{2n}$, $\omega(v, w)$ is $v^T matrix w$.
$$J_0(\mathbb{R}^{2n,\omega_0} = \{j \in \mathrm{End}(\mathbb{R}^{2n}) \mid$$
   - $J^2 = -\mathrm{Id}$ (a.c.s.)
   - $J^T J_0 J = J_0$ (J is symplectic)
   - $J_0 J$ is positive definite}

   Bullet points two and three together is equivcalent to $\omega_0(v, Jw)$ is a metric.

   Aside: Hermitian form on $\mathbb{R}^{2n} = \mathbb{C}^n$:
$$H(v, w) = \overline{v}^T w.$$
   Unitary group
$$U(n) = \{A \in \mathrm{End}(\mathbb{R}^{2n}) \mid A^* H = H\}$$
   where $A^*H = H$ is equivalent to
$$H(Av, Aw) = H(v, w) \forall v, w.$$

Fact. a.c.s. + symp $\Rightarrow J_0 J$ is symmetric, i.e. $(J_0 J)^T = J_0 J$. Also symplectic because symplectic matrices form a group.

So we have a map

$$J_0(\mathbb{R}^{2n}, \omega_0) \to \mathcal{P}(\mathbb{R}^{2n}, \omega_0) = \{\text{symmetric positive definite symplectic matrices}\}$$

$$J \mapsto J_0 J$$

$$-J_0 P \leftarrow\!\shortmid P.$$

We can check this is one-to-one correspondence. In face it is a diffeomorphism.

Goal now: To show $\mathcal{P}(\mathbb{R}^{2n}, \omega_0)$ is contractible. The reason is that $\mathcal{P}(\mathbb{R}^{2n}, \omega_0) \subset \mathcal{P}(\mathbb{R}^{2n})$, the set of positive definite symmetric matrices. Then contractivle contraction preseves $\mathcal{P}(\mathbb{R}^{2n}, \omega_0)$.

$P \in \mathcal{P}(\mathbb{R}^{2n})$, diagonalize:

$$P = S^{-1} \cdot diag(\lambda_1, \ldots, \lambda_{2n}) \cdot S$$

where $\lambda_i \in \mathbb{R}_{\geq 0}$, $S \in \mathrm{GL}(2n, \mathbb{R})$.

For $\alpha \geq 0$, define

$$P^\alpha = S^{-1} \cdot diag(\lambda_1^\alpha, \ldots, \lambda_{2n}^\alpha) \cdot S.$$

Check: independent of diagonalization. Now

$$\mathcal{P}^\alpha \in \mathcal{P}(\mathbb{R}^{2n}) \forall \alpha.$$

Define

$$f_t \colon \mathcal{P}(\mathbb{R}^{2n}) \to \mathcal{P}(\mathbb{R}^{2n})$$

via

$$f_t(P) = P^t, 0 \leq t \leq 1.$$

This is a deformation retract of $\mathcal{P}(\mathbb{R}^{2n})$ to the identity. FOr the "contractible contraction preserves $\mathcal{P}(\mathbb{R}^{2n}, \omega_0)$" see McDuff Salamon.

Step 2. $\mathrm{End}(TM)$ is a smooth vector bundle over M, where the fiber over each point $x \in M$ is $\mathrm{End}(T_x M)$. There is the space

$$J_x(M, \omega) \subset \mathrm{End}(T_x M)$$

by definition.

Define

$$J'(M, \omega) = \bigcup_{x \in M} J_x(M, \omega) \subset \mathrm{End}(TM).$$

There is the natural projection map

$$J'(M, \omega) \to M.$$

Claim. This is a fiber bundle.

Proof. Need to checl local triviality. Reason: Darboux's Theorem.

Given $x \in M$, choose a neighborhood V of x such that there is a symplectomorphism

$$\phi\colon (V,\omega) \to (U,\omega_0)$$

where $U \subset \mathbb{R}^{2n}$ is some open domain. And we assume x is mapped to 0. Then we have a commutative square:

$$\begin{array}{ccc} \pi^{-1}(V) & \longrightarrow & J_0 \\ \downarrow & & \downarrow \\ V & & U \end{array}$$

Step. 3 Need to show $J(M,\omega)$ is the space of smooth sections of $J'(M,\omega)$, which is a smooth fiber bundle with non-empty contractible fibers. Then it follows that $J(M,\omega)$ is non-empty and contractible.

Exercise. Check for a trivial fiber bundle with contractible fibers.

Exercise. Prove it in general or see either Steenrod or Mathoverflow.

Recall we were studying

$$u\colon \mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to M$$

such that

$$\partial_s u + J_t \cdot (\partial_t U - X_{H_t} \circ u) = 0$$

this is CRF equation. Here we are asuming (M,ω) is a copact sympelctic manifold, J_t are compatilbe a.c.s., and H_t time-dependent Hamiltonians.

Define

$$\tilde{u}(s,T) = (\phi^T_{H_t})^{-1}$$

Lecture 23: 20 Mar

Solutions to u to CRF are equivalent to solutions

$$\tilde{u}\colon \mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to M$$

to

$$\partial_s \tilde{u} + \tilde{J}_t(\partial_t \tilde{u}) = 0$$

for a suitalbe transformation

$$C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, M) \to C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, M)$$
$$u \mapsto \tilde{u}$$

and of

$$J_t \mapsto \tilde{J}_t.$$

Define

$$\tilde{u}(s,t) = (\phi^t_{H_t})^{-1} \circ u(s,t)$$

and

$$\tilde{J}_t = ((\phi^t_{H_t})^{-1})^* J_t = (\mathrm{d}\phi^t_{H_t})^{-1} J_t (\mathrm{d}\phi^t_{H_t}).$$

Note. Even if J_t is time-independent, this $\tilde{J}_t$ is time-dependent.

Proof of this equivalency is just a computation:

Proof.

$$\begin{aligned} \partial_s u &= \partial_s (\phi_{H_t})^t \circ \tilde{u}) && (4.14) \\ &= \mathrm{d}\phi^t_{H_t}(\partial_s \tilde{u}) && (4.15) \end{aligned}$$

$$\begin{aligned} \partial_t u &= \partial_t (\phi^t_{H_t} \circ \tilde{u}) && (4.16) \\ &= \mathrm{d}\phi^t_{H_t}(\partial_t \tilde{u}) + (X_{H_t} \circ \phi^t_{H_t}) \circ \tilde{u} && (4.17) \end{aligned}$$

Thus

4.2 Pseudoholomorphic Curves

The cylinder $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ is the quotient of $\mathbb{C}$ by vertical integer translation. So the cylinder inherits the a.c.s. j from $\mathbb{C}$:

$$j\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t}$$

this implies

$$j\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial s}.$$

Suppose

$$u\colon \mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to (M, J)$$

where J is a a.c.s. Then u obeys CR:

$$\partial_s u + J(\partial_t u) = 0$$

if and only if u intertwines the two a.c.s. (j, J), i.e.

$$u\colon \left(\mathbb{R} \times {}^{\mathbb{R}}/_{\mathbb{Z}} \to (M, J), j\right)$$

of almost complex manifolds.

A motphism etween almost complex manifolds

$$f\colon (M_1, J_1) \to (M_2, J_2)$$

is a smooth map obeying

$$J_2 \circ \mathrm{d}f = \mathrm{d}f \circ J_1$$

(f intertwines J_1 and J_2).

Proof. Suppose $u\colon \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$.

$$\mathrm{d}u \circ j\left(-\frac{\partial}{\partial s}\right) = \mathrm{d}u\left(\frac{\partial}{\partial t}\right) = -\partial_s u$$

$$J \circ \mathrm{d}u\left(\frac{\partial}{\partial t}\right) = J(\partial_t u)$$

Hence

$$\partial_s u + J(\partial_t u) = 0$$

if and only if

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u$$

If $\dim M_1 > 2$, then generically, there are no non-constant morphisms to $(\mathbb{C}, j)$. If $\dim M_1 = 2$, then the situation is much different:

If (Σ, j) is a 2-dim almost complex manifold, then it is *integrable*, i.e. there exists a complex structure on Σ whose affiliated a.c.s. is j (fundamental Thm. about Riemann surfaces).

If (M, J) is any almost complex manifold, one expects lots of morphisms

$$(\Sigma, j) \to (M, J).$$

To wit: if $x \in M$, and $0 \neq v_x \in T_x M$, then there exists a map

$$f\colon (B_\varepsilon(0), j) \to (M, J)$$

where $B_\varepsilon(0) \subset \mathbb{C}$, and such that

$$f(0) = x, \quad \mathrm{d}f((1,0)_0) = v_x.$$

We call a morphism

$$(\Sigma, j) \to (M, J)$$

a *pseudoholomorphic curve* or *J-holomorphic curve.*

Regularity: If (M, J) is a smooth almost complex manifold, and

$$u\colon (\mathbb{C} \supset U, j) \to (M, J)$$

is once-differentiable, and is furthermore J-holomorphic, then u is in fact smooth. (something about elliptic regularity)

4.2.1 Area and Energy of J-holomorphic Curves

Warm-up: Suppose

$$u\colon (U, j) \to (M, J)$$

where $U \subset \mathbb{C}$. To calculate area of u, we want to integrate

$$g(\partial_s u, \partial_t u)\mathrm{d}s\mathrm{d}t$$

over U, where g is the standard metric on $\mathbb{C}^n$

Lecture 23: 22 Mar

4.3 Area and Energy of Pseudoholomorphic Curves

Area: if $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, we are interested in the k-dim volume of the parallelopiped they span is gotten by

) (

Gram matrix, and then taking the square root of the determinant.

Suppose $U \subset \mathbb{R}^2$ with coordinates (s, t), and

$$f \colon U \to (M, g)$$

where (M, g) is a Riemannian manifold. Then

$$Area(f) = \int_U \det \begin{pmatrix} g(\partial_s f, \partial_s f) & g(\partial_s f, \partial_t f) \\ g(\partial_t f, \partial_s f) & g(\partial_t f, \partial_t f) \end{pmatrix}^{1/2} \mathrm{d}s\mathrm{d}t$$

This is a diffeomorphism invariant, i.e. if

$$\phi \colon \mathbb{R}^2 \supset V \to U$$

is a diffeo, then

$$Area(f \circ \phi) = Area(f).$$

Then if Σ is a smooth surface and

$$f \colon \Sigma \to (M, g)$$

is a smooth map, then we can define $Area(f)$ using the above definition on local coordinates.

Energy: We ahve $f, U, (M, g)$ as before, (here $|\partial_s f|^2$ is shorthand for $g(\partial_s f, \partial_s f)$)

$$energy(f) = \frac{1}{2} \int \left(|\partial_s f|^2 + |\partial_t f|^2 \right) \mathrm{d}s\mathrm{d}t$$

Compare: if $\gamma \colon [a, b] \to \mathbb{R}^n$, then

$$energy(\gamma) = \frac{1}{2} \int_a^b |\partial_t \gamma|^2 \mathrm{d}t$$

integral of $\frac{1}{2}|\partial_t \gamma|^2$ is the kinetic energy of a particle of mass 1.

This came up earlier: if $\gamma \colon \mathbb{R} \to (M, g)$ is a flow line of $-\nabla_g f$ for f a Morse function, then

$$energy(\gamma) = f(p_-) - f(p_+)$$

where

$$p_\pm = \lim_{t \to \pm\infty} \gamma(t).$$

Caution: energy is not a diffeomorphism invariant, but it is a conformal invariant: treat $\mathbb{R}^2 = \mathbb{C}$ where $(s, t) \in \mathbb{R}^2$ corresponds to $s + it \in \mathbb{C}$. If we have

$$\phi \colon \mathbb{C} \supset V \to U \subset \mathbb{C}$$

which is conformal, i.e. a bholomorhism ie.e a diffeomorphism satisfying

$$j \circ \mathrm{d}\phi = \mathrm{d}\phi \circ j.$$

then

$$energy(f \circ \phi) = energy(f).$$

If (Σ, j) is a Riemann surface, then we can define $energy(f)$ by locally by our energy formula, for any smooth map

$$f : (\Sigma, j) \to (M, g).$$

So now, let's suppose (M, ω) is a symplectic manifold, we choose a compatible a.c.s. $J(M, \omega)$, and g the affiliated metric, i.e. $g(v, w) = \omega(v, Jw)$ for all $x \in M$, $v, w \in T_x M$. If we have a map from a Riemann surface to M,

$$f : (\Sigma, j) \to (M, \omega)$$

obeying

$$\mathrm{d}f \circ j = J \circ \mathrm{d}f.$$

then we call f (j, J)-holomorphic, and $\mathrm{Im}(f)$ is called a J-holomorphic or a pseudoholomorphic curve.

Proposition 16. Suppose

$$f : (\Sigma, j) \to (M, \omega, J)$$

is a smooth map. Then

$$Area(f) \geq \int_\Sigma f^*\omega$$

and

$$energy(f) \geq \int_\Sigma f^*\omega.$$

Equality holds in the first one if and only if $\mathrm{Im}(f)$ is a pseudoholomorphic curve; and equality holds in the second one if and only if f is J-holomoprhic (which would imply $\mathrm{Im}(f)$ is a pseudoholomoprhic curve).

If Σ is closed, and we let $A := f_*[\Sigma] \in H_2(M)$, then the RHS is

$$\int_\Sigma f^*\omega = [\omega](A),$$

which is a topological invariant; whereas $Area(f)$ is a geometric one, and $energy(f)$ is an analytic one.

Aside. non-constant connected psudoholomorphic curve $\equiv$ immersed surface whose tangent planes are all J-holomoprhic "lines" in (M, J). This requires a theorem along the lines of analytic continuation. for J-holomorphic curves.

Slogan: Pseudoholomorphic curves minimize

Corollary. A pair of psh curve representing $A \in H_2(M)$ have the same area; and J-holomorphic parametrizations of them have the same energy, and both area and energy are equal to $[\omega](A)$.

In an arbitrary (M, J) where J is not compatible with some symplectic form, a one-parameter family of J-holomorphic curves $(\Sigma, j) \to (M, J)$ could badly degenerate, have bits, "degenerate". So no good deformation theory. But with J compatible with some ω, the energy control we get will allow us to undestand deformations, which is the idea of Gromov compactness.

Lecture 25: 24 Mar

Want to prove the Proposition. Note that the quantity on the RHS of the inequalities are independent of J.

Proof. We start with proving the second inequality

$$energy(f) \geq [\omega](A).$$

Recall, in a chart U of Σ, with complex coordinates $s + it$. We have

$$energy(f\Big|_U) = \frac{1}{2}\int_U (|\partial_s f|_g^2 + |\partial_t f|_g^2)\mathrm{d}s\mathrm{d}t.$$

Note that J is an isometry of g; indeed, for all $x \in M$ and $v, w \in T_x M$, we have

$$\begin{aligned} g(Jv, Jw) &= \omega(Jv, J(Jw)) && (4.18)\\ &= \omega(Jv, -w) && (4.19)\\ &= \omega(w, Jv) && (4.20)\\ &= g(w, v) && (4.21)\\ &= g(v, w). && (4.22)\end{aligned}$$

Thus,

$$\begin{aligned} |\partial_s f|^2 + |\partial_t|^2 &= |_s f|^2 + |J\partial_t f|^2 && (4.23)\\ &= |\partial_s f + J\partial_t f|^2 - 2g(\partial_s f, J\partial_t f) && (4.24)\\ &= |\partial_d f + J\partial_t f|^2 - 2\omega(\partial_s f, J(J\partial_t f)) && (4.25)\\ &= |\partial_s f + J\partial_t f|^2 + 2\omega(\partial_s f, \partial_t f) && (4.26)\end{aligned}$$

The left term is ≥ 0, and equality iff. CR satisfied.

Thus,

$$energy(f\Big|_U) \geq \qquad (4.27)$$

□

Preview of Gromov compactness Suppose

$$u_n\colon (\Sigma, j) \to M$$

are all J-holomorphic and

$$energy(u_n) \le E_0$$

i.e. bounded energy. Important case: if u_n are J-holomorphic and they all represent the same homology class:

$$u_n[\Sigma] = A \in H_2.$$

Then one possibility is: for all (s,t) (in a holomorpphic chasrt),

$$|\partial_s u_n(s,t)|^2 + |\partial_t u_n(s,t)|^2$$

stays bounded, i.e. there exists a uniform bound on all pointwise energies. Then Arzela-Ascoli implies that there exsits a subsequence u_n which converges uniformly in C^1 to a (once differentiable) map $u\colon \Sigma \to M$, and u will be J-holomorphic.

Lecture 27: 29 Mar

Set-up: (M,ω) is a closed symplectic manifold.

Let Σ denote a closed, smooth surface (no a.c.s. on Σ yet). Let Γ be a set of pairwise disjoint simple closed curves on Σ. Let Σ_0 be the surface obtained by the one-point compactification of each end of $\Sigma - \Gamma$ (the end-compactification points pair up into nodal pairs).

Let $\overline{\Sigma}$ be the space obtained by collapsing each loop in $\Gamma \subset \Sigma$ to a point, this is equivalent to indetigying points in the same nodal pair (nodes are as individual–??)

We have quotient maps

$$q\colon \Sigma \to \overline{\Sigma}$$
$$q_0\colon \Sigma_0 \to \overline{\Sigma}$$

By definition, an a.c.s. $\overline{J}$ on $\overline{\Sigma}$ is an a.c.s. on Σ_0.

A *parametrized cusp curve* is a map

$$\overline{u}\colon (\overline{\Sigma}, \overline{J}) \to (M, J)$$

such that

$$\overline{u} \circ q_0\colon (\Sigma_0, \overline{J}) \to (M, J)$$

is (j,J)-holomorphic away from the nodal pairs, and such that $q \circ u$ is continuous.

Now suppose that j is an a.c.s. on Σ and a sequence

$$u_n\colon (\Sigma, j) \to (M, J)$$

that are all J-holomorphic.

Definition 16. We say u_n weakly converges to $\overline{u}$ if there exist diffeomorphisms ϕ_n of Σ such that

1. $u_n \circ \phi_n$ converges C^0 (uniformly and continuously) to $\overline{u} \circ q$
2. $u_n \circ phi_n$ converges in C^∞_{loc} to $\overline{u} \circ q$ on $\Sigma - \Gamma$

3. $(\phi_n)_* j$ converges in C^∞_{loc} to $\overline{J}$ on $\Sigma - \Gamma = \Sigma_0 \setminus$ nodal pairs

Recall C^∞_{loc}-convergence means for all compact subset $K \subset \Sigma - \Gamma$, $u_n \circ \phi_n\big|_K$ converges in C^∞ to $\overline{u} \circ q$ on K.

Theorem 12 (Basic form of Gromov compactness). If $u_n \colon (\Sigma, j) \to (M, J)$ is a sequence of J-holomorphic maps with bounded energy:

$$evergy(u) = Area(u_n) = [\omega](u_n)_*[\Sigma] \leq E_0 < \infty$$

then there exists a subsequence of u_n which weakly converges to a cusp curve.

Link this with the example

$$u_r \colon [x \colon y] \mapsto [x^2 \colon y^2 \colon rxy]$$

from last class.

Addendum: A J-holomoprhic map $u \colon (\Sigma, j) \to (M, J)$ is called *multiply-covered* if there exists a branched covering map

$$\pi \colon (\Sigma, j) \to (\Sigma', j')$$

of degree > 1 and a

$$v \colon (\Sigma', j') \to (M, J)$$

such that

$$u = v \circ \pi.$$

Baby example:

$$u_0 \colon \mathbb{C}P^1 \to \mathbb{C}P^2$$

is multiply-covered since

$$\pi \colon \mathbb{C}P^1 \to \mathbb{C}P^1$$

with

$$\pi([x \colon y]) = [x^2 \colon y^2]$$

and

$$v \colon \mathbb{C}P^1 \to \mathbb{C}P^2$$

$$v([x \colon y]) = [x \colon y \colon 0]$$

gives

$$u_0 = \pi \circ v.$$

If u is not multiply-covered, then it is *simple*. All cylinders in Floer homology are simple.

Simple test: u is simple if and only if it is *somewhere-injective*, which means there exists $z_0 \in \Sigma$ such that $u(z) = u(z_0)$ if and only if $z = z_0$, and $\mathrm{d}u_{z_0} \neq 0$. If u is simple and Σ is conneted, then the set of somewhere injective points is open and dense in the domain.

The conclusion ofthe Addendum: if u_n are all simple, then so is the weak Gromov limit $\overline{u}$, unless $\overline{\Sigma}$ is itself a Riemann surface.

Suppose (M, ω) is a closed symplectic manifold, $J \in J(M, \omega)$ a compatible a.c.s.

The tangent bundle, equipped with J (TM, J) is a complex vector bundle over M, which gives rise to

$$c_1(TM, J) \in H^2(M)$$

$J(M, \omega)$ is contractible, so any two choices $J, J' \in J(M, \omega)$ give rise to homotopic complex vector bundles $(TM, J), (TM, J')$, thus

$$c_1(TM, J) = c_1(TM, J')$$

thus we can unambiguouslky define

$$c_1(M, \omega) := c_1(TM, J)$$

for any $J \in J(M, \omega)$. (McDuff-Salomon)

Select $A \in H_2(M)$. Fix a Riemann surface (Σ, j)

Define

$$\mathcal{M}(A, J) := \{J\text{-hol maps } u\colon (\Sigma, j) \to (M, J) \mid u_*[\Sigma] = A, u \text{ simple}\}$$

For instance, if $(M, \omega, J) = (\mathbb{C}P^2, \omega_{std}, J_{std})$, $(\Sigma, j) = (\mathbb{C}P^1, j)$, and

$$A = 2H[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$$

then

$$u_r \in \mathcal{M}(A, J)$$

fro all $0 < r < \infty$.

Theorem 13 (Gromov). For a generic choice of $J \in J(M, \omega)$, $\mathcal{M}(A, J)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(A, J) = \frac{1}{2}(\dim M) \cdot \chi(\Sigma) + 2c_1(M, \omega) \cdot A$$

So in our example,

Lecture 28: 3 Apr

Today: consequences of Gromov compactness. (See Gromov's paper on pseudoholomorphic curves)

As usual, (M, ω) a closed symplectic manifold, $J(M, \omega)$ the space of compatible a.c.s. on (M, ω).

Gromov compactness states: if $u_n\colon (\Sigma, j) \to (M, J)$, $J \in J(M, \omega)$ is a sequence of parametrized psudoholomorphic curves, and there is an absolute bound on the energies:

$$E(u_n) \leq E_0 < \infty$$

then there exists a subsequence which weakly converges to a parametrized cuspidal curve.

pic

Special case: if all of these maps represent a fixed homology class:

$$(u_n)_*[\Sigma] = A,$$

$A \in H_2(M)$ independent of n, then

$$E(u_n) = \omega(A) \quad \forall n$$

and the cuspidal curve u has

$$E(u) = \omega(A),$$

and

$$u_*[\Sigma] = A.$$

And

$$u_*[\Sigma] = \sum_{\Sigma_j \text{ component of } \Sigma} u_*[\Sigma_j]$$

similarly

$$E(u) = \sum u(\Sigma_j)$$

Genericity/dimension theorem

$$\mathcal{M}(A, J) \coloneqq u\,\{u\colon (\Sigma, j) \to (M, J) \mid u \text{ } J\text{-holomorphic}, u_*[\Sigma] = A, u \text{ simple}\}$$

For generic J, this space $M(A, J)$ is a smooth manifold of dimension

$$\frac{1}{2}(\dim(M)) \cdot \chi(\Sigma) + 2c_1(A)$$

here $c_1(M, \omega) \coloneqq c_1(M, J)$.

> **Remark.** Genericity: transversality in symplectic geomoetry, involving $\overline{\partial}$ operator, elliptic regularity. Fredholm theory, Sard-Smale regularity for Banach manifolds.
>
> The dimension formula comes from Riemann-Roch Theorem.
>
> (See McDuff-Salamon: J-holomorphic curves and Quantum Cohomology)

Suppose

$$\inf\{\omega(B) > 0 \mid B \in H_2(M)\} > 0$$

PAUSE

> **Example.** $(M, \omega) = (\mathbb{C}P^2, \omega_{FS})$, and take $(\Sigma, j) = (\mathbb{C}P^1, J_{std})$, and $A = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$.
>
> $\mathcal{M}(A, J)$ (for generic J) is a smooth manifold of dimension $\frac{1}{2} \cdot 4 \cdot 2 + 2 \cdot 3 = 10$.
>
> Take J to be the standard J on $\mathbb{C}P^2$, this is saying the space of parametrized lines inside $\mathbb{C}P^2$ is of dimension 10. Let's check this:
>
> Any pair of points in $\mathbb{C}P^2$ determines a unique $\mathbb{C}P^1$: this seems to give $4+4$ dimensions ofdistinct pairs of points in $\mathbb{C}P^2$; but we need to substract the points that give us the same $\mathbb{C}P^1$. We subtract $(2 + 2)$ dimensions of points on $\mathbb{C}P^1$. So we get a 4-dimensional space of unparametrized $\mathbb{C}P^1$'s inside $\mathbb{C}P^2$.
>
> There is a 6-dimensional family of parametrizitons of each $\mathbb{C}P^1$, because

the space of automorphisms,

$$\mathrm{Aut}(\mathbb{C}P^1, j_{std})$$

is 6-dimensional (this is the space of Mobius transformations i.e. $PSL(2,\mathbb{C})$
So we have

$$10 = 4 + 6$$

In fact, any $J \in J(\mathbb{C}P^2, \omega_{FS})$ is *regular*, in the sense that $\mathcal{M}(A, J)$ is a smooth manifold of the correct dimension. So we get a four-dimensional family of unparametrized J-holomoprhic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$ for any $J \in J(\mathbb{C}P^2, \omega_{FS})$

Addendum to Gromov compactness: $G = \mathrm{Aut}(\Sigma, j)$ acts on $\mathcal{M}(A, J)$ by precomposition. Note: $\mathrm{Aut}(\mathbb{C}P^1, j_{std})$ is six-dimensional,

pic

Theorem 14. For generic J,

$$\mathcal{M}(A, J)\,/G$$

admits a compactification by the space of cusp curves representing (A, J). And $\mathcal{M}(A, J)/G$ is a compact manifold of dimsneion $\frac{1}{2}\dim M \cdot \chi(\Sigma) + 2c_1(A) - \dim G$ (**).

PLAY
Suppose

$$\inf\{\omega(B) > 0 \mid B \in H_2(M)\} > 0$$

this happens for instance:

1. $(M, \omega) = (\mathbb{C}P^2, \omega_{FS})$
2. $(M, \omega) = (S^2 \times S^2, \omega_{FS} \oplus \omega_{FS})$
3. Non-example: $(M, \omega) = (S^2 \times S^2, \omega_{FS}\lambda\omega_{FS})$ where $\lambda \notin \mathbb{Z}$. Here the infimum is 0.

Assume this happens, pick a class $A \in H_2(M)$ such that $\omega(A) > 0$ Then any cusp curve which represents A has just one component. Morever, it cannot be multiply-covered. Hence it is an honest simple J-holomoprhic curve that represents A. Therefore, $\mathcal{M}(A, J)/G$ is a closed manifold of the expected dimension (**).

Let us focus on the case $(\mathbb{C}P^2, \omega_{FS})$. For any $j \in J(\mathbb{C}P^2, \omega_{FS})$, $\mathcal{M}(A, j)/G$ is a compact four-dim space. Moreover, for any pair of distinct points in $\mathbb{C}P^2$, there exists a unique J-holomorphic curve passing through them. For any fixed point $p \in \mathbb{C}P^2$, the space of J-holomorphic genus-zero curves passing through p give a singular foliation of $\mathbb{C}P^2$.

Now to the case of $(M, \omega) = (S^2 \times S^2, \omega_{FS}\omega_{FS})$. Take

$$A = [S^2 \times \star]$$

and

$$B = [\star \times S^2].$$

As it turns out, for any $J \in J(M, \omega)$ you choose, the set of J-hol genus-0 curves representing A foliate $S^2 \times S^2$, as fo the J-hol genus-0 curves representing B and any A curve intersects any B-curve in a unique point.

Thisfeeds into Gromov's $\mathbb{R}^4$ recegininition theorem:

Lecture 29: 12 Apr

From pseudoholomorphic curves to Floer homology.

Set-up for Floer homology: (M, ω) a closed symplectic manifold,

$$H = H_t \in C^\infty(M \times \mathbb{R}/\mathbb{Z})$$

1-periodic time-dependent Hamiltonian.

We also assume $\pi_2(M) = 0$. And assume all fixed points of $\phi^1_{H_t}$ are non-degen.

THen CF is a chain complex freely generated over $\mathbb{F}_2$ by contractible periodic orbits of Hamiltonian flow:

$$\gamma \colon \mathbb{R}/\mathbb{Z} \to M$$

- contractible, i.e. $\gamma \in \mathcal{L}(M)$
- $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_{H_t} \circ \gamma$

We saw that such $\gamma \in \mathcal{L}(M)$ are critical points of the symplectic action funtional

$$\mathcal{A} = \mathcal{A}_H = \mathcal{A}_{H_t}$$

where

$$\mathcal{A}(\gamma) = \int_0^1 H_t \circ \gamma(t)\mathrm{d}t + \int_{D^2} \hat{\gamma}^*\omega.$$

We study the gradient flow of $\mathcal{A}$, after first choosing a compatible $J \in J(M, \omega)$; or, with no added complication, $J_t \in J(M, \omega)$, $0 \le t \le 1$, smooth in t, time-dependent, i.e. we can write

$$J_t \in C^\infty([0,1], J(M, \omega))$$

such that

$$u \colon \mathbb{R} \to \mathcal{L}(M)$$

$$\frac{\mathrm{d}u}{\mathrm{d}s} + \nabla_{J_t}\mathcal{A}_{H_t} \circ u = 0$$

we unraveled this equation into the form

$$u \colon (s,t) \ni \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \to M$$

$$\partial_s u + J_t(\partial_t u - X_{H_t} \circ u) = 0$$

the Cauchy-Riemann-Fleor equation.

Reparamization trick: Define

$$\tilde{u}(s,t) = (\phi_H^t)^{-1} \circ u(s,t)$$

$$\tilde{J}_t = \mathrm{d}{\phi_H^t}^{-1} J_t(\mathrm{d}\phi_H^t), 0 \le t \le 1$$

then CRF holds iff

$$\partial_s \tilde{u} + \tilde{J}_t(\partial_t \tilde{u}) = 0.$$

TFAE: given u solving CRF,

1. $\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t)$ contractivle closed Hamiltonian orbits, convergence unitform in t as $s \to +\infty$ or $s \to -\infty$.

2. $\lim_{s\to\pm\infty} \tilde{u}(s,t) = X_\pm \in Fix(\phi_H^1)$ converge uniform in t

3. $$E(\tilde{u}) = \frac{1}{2}\int_{\mathbb{R}\times(\mathbb{R}/\mathbb{Z})} (|\partial_s \tilde{u}|^2 + |\partial_t \overline{u}|^2)\mathrm{d}s\mathrm{d}t < \infty$$

   where

   $$|\cdot| = g_t(\cdot,\cdot) = \omega(\cdot, \tilde{J}_t \cdot)$$

4. $$E(u) = \frac{1}{2}_{\mathbb{R}\times(\mathbb{R}/\mathbb{Z})} (|\partial_s u|_t^2 + |\partial_t u - X_{H_t} \circ u|_t^2)\mathrm{d}s\mathrm{d}t < \infty$$

   where

   $$|\cdot|_t = g_t(\cdot,\cdot) = \omega(, J_t)$$

Exercise.

$$E(u) = \mathcal{A}(\gamma_-) - \mathcal{A}(\gamma_+)$$

Following the exercise, fix $\gamma_\pm \in Orb(H_t)$ and define

$$\mathcal{M}(\gamma_-, \gamma_+, J_t, H_t) \left\{ u \text{ solving CRF s.t. } \lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t) \text{ conv unif in } t \right\}$$

> **Remark.** Every $u \in \mathcal{M}(\gamma_-, \gamma_+, J_t, H_t)$ has
>
> $$E(u) = \mathcal{A}(\gamma_-) - \mathcal{A}(\gamma_+).$$

Floer proved a relative version of Gromov's compactness theorem for solutions to CRF. Specifically, if u_n is a sequence of solutions to CRF, and moreover the energies remain bounded: $E(u) \le E_0 < \infty$ for all n, then u_n has a subsequence which weakly converges to a "cusp cylinder". We recycle notation such that u_n denotes the subsequence. Heuristically,