KHOVANOV HOMOLOGY, KNOT FLOER HOMOLOGY, AND THE (2,7) TORUS KNOT

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This paper is dedicated to Melody, Tor, and Gary.

Abstract. We prove that Khovanov homology, together with knot Floer homology, both with coefficients in $\mathbb{Z}/2\mathbb{Z}$, detects the $(2,7)$ torus knot, assuming the knot in question has symmetric monodromy. Our proof follows the techniques used in [BHS25], in which the result is proved for the $(2,5)$ torus knot, while only assuming the knots have identical Khovanov homology. The technique involves combining tools from knot homology theories with classical results on the dynamics of surface homeomorphisms, culminating in reducing the detection question to a problem about mutually braided unknots. As in the aforementioned paper, we employ computer assistance to solve this problem, though our computational requirements are substantially more intensive.

[1].

1. Introduction

We prove that Khovanov homology, together with knot Floer homology detects the torus knot $T(2,7)$, assuming the fibering monodromy of the knot is *symmetric*, i.e. commutes with a hyperelliptic involution. The following is our main result:

Theorem 1.1. *Suppose $K \subset S^3$ is a knot whose reduced Khovanov homology over $\mathbb{Z}/2\mathbb{Z}$ is 7-dimensional and is supported in a single positive δ-grading, and suppose*

$$\widehat{HFK}(K) \cong \widehat{HFK}(T(2,7))$$

as graded vector spaces over $\mathbb{Z}/2\mathbb{Z}$. Suppose in addition that there exists a hyperelliptic involution τ such that the fibering monodromy of K commutes with τ. Then $K = T(2,7)$.

1.1. Outline.

1.2. Acknowledgments.

2. Pseudo-Anosov Maps

In this section, we provide a very brief review of some basic facts and terminology related to pseudo-Anosov homeomorphisms of surfaces.

Suppose

$$h \colon \Sigma \to \Sigma$$

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[1]Here is an example of a footnote. Notice that this footnote text is running on so that it can stand as an example of how a footnote with separate paragraphs should be written.

And here is the beginning of the second paragraph.

is a homeomorphism of a compact, orientable surface Σ with (possibly empty) boundary boundary and/or with marked points. The Nielsen–Thurston classification ([Thu88]) states that h is freely isotopic (i.e. not required to be the identity on $\partial\Sigma$) to a homeomorphism ϕ_h which is either:

- periodic, meaning that $\phi_h^n = id$ for some positive integer n;
- reducible, meaning that there is a non-empty set $c = c_1, \ldots, c_n$ of disjoint, essential, simple closed curves in Σ such that $\{\phi_h(c_i)\}_{i=1}^n = c$; or
- pseudo-Anosov

We call ϕ_h a *geometric representative* of h. We will focus on the case in which ϕ is pseudo-Anosov. In this case it is known that it is neither periodic nor reducible.

The map ϕ is *pseudo-Anosov* if there exists a constant $\lambda > 1$ and a pair of transverse, measured, singular foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ (collectively termed *invariant foliations*) such that:

- $\phi(\mathcal{F}_u, \mu_u) = (F_u, \lambda_{\mu_u})$
- $\phi(\mathcal{F}_s, \mu_s) = (F_s, \lambda_{\mu_s}^{-1})$

and such that the singularity types are subject to the following constraints: non-marked interior singular points have at least 3 prongs, and boundary components consist of some number $k \geq 1$ of 1-pronged singularities, which we often think of collectively as a "k-pronged boundary".

The number λ is called the *dilatation* of the pseudo-Anosov ψ, and is a topic of interest in its own right.

Suppose Σ has exactly one boundary component. If $\mathcal{F}_s$ and $\mathcal{F}_u$ meet $\partial\Sigma$ in $n \geq 2$ prongs, then ϕ extends naturally to a pseudo-Anosov homeomorphism

$$\hat{\phi}\colon \hat{\Sigma} \to \hat{\Sigma}$$

of the closed surface $\hat{\Sigma}$ obtained from Σ by capping off the boundary with a disk. Moreover, the invariant foliations of ϕ extend to stable and unstable foliations $\hat{\mathcal{F}}_s$ and $\hat{\mathcal{F}}_u$ for $\hat{\phi}$ in which the n-prongs on $\partial\Sigma$ extend to n-prongs meeting at a singularity p in the capping disk (except that p is a smooth point when $n = 2$). Note that p is a fixed point of $\hat{\phi}$.

Suppose still that Σ has a single boundary component. One dynamical aspect of ϕ which will be important is the *fractional Dehn twist coefficient* $c(h)$. Roughly, $c(h)$ measures how the geometric representative ϕ behaves near $\partial\Sigma$. When ϕ is pseudo-Anosov, we use the definition $c(h) := n + m/k$, where h acts as n full twists near $\partial\Sigma$; the invariant foliations of ϕ have k singular points on $\partial\Sigma$; and ϕ acts as a m/k rotation on the cyclically-ordered set of singular points on $\partial\Sigma$. Note that when the invariant foliations of ϕ have a single prong on $\partial\Sigma$, we have $c(h) \in \mathbb{Z}$, since there is no "fractional part." Conversely, if $c(h) \in \mathbb{Z}$ then we can conclude that ϕ does not rotate the boundary singularity.

A *hyperelliptic involution* is an order-two element of $\mathrm{Mod}(\Sigma_g)$ that acts by $-I$ on $H_1(\Sigma_g; \mathbb{Z})$. It is well-known that for $g = 2$, there is a unique hyperelliptic involution. Furthermore, every homeomorphism of the closed genus-two surface is isotopic to some homeomoprhism that commutes with this unique hyperelliptic involution. For $g \geq 3$, this uniqueness fails, and commutativity can fail as well. We will mostly restrict our attention to pseudo-Anosov homeomorphisms ϕ on the genus three surface which are isotopic to some map that commutes with *some* hyperelliptic involution on the closed genus three surface. We will call such maps *symmetric*. In general, we will also extend this terminology to maps on a surface with possibly non-empty boundary.

3. Floer homology and fixed points

The purpose of this section is to present some information that will be needed to prove

It is well-known from the work of Ozvath and Szabo [OS04] that knot Floer homology detects fiberedness and the genus of the knot, and that this information is contained in the top Alexander grading. The work of Ni and Yi [Ni22] shows that for a fibered knot, the second-to-top Alexander grading contains information regarding the number of fixed points of the fibering monodromy:

Theorem 3.1 ([Ni22] Theorem 1.2)**.** *Let Y be a closed, oriented 3–manifold, and $K \subset Y$ be a fibered knot with fiber F and monodromy φ. If*

$$rank\widehat{HFK}(Y, K, [F], g(F) - 1; \mathbb{F}) = r$$

then φ is freely isotopic to a diffeomorphism with at most $r - 1$ fixed points.

Particular to our case, this implies that a knot K with the same knot Fleor homology as that of $T(2, 7)$ necessarily must have no fixed points.

The following Lemma establishes that if there were a knot K with identical Alexander polynomial to that of $T(2, n)$ (in particular this would be the case if the knot Floer homologies were identical), and if one assumes $K \neq T(2, n)$, then K must be hyperbolic. The argument follows almost identically to that of Lemma 3.2 of [BHS25], where the analogous fact is proved specifically for $T(2, 5)$.

Lemma 3.2. *Let $K \subset S^3$ be a fibered knot with Alexander polynomial identical to that of the torus knot $T(2, n)$, that is:*

$$\Delta_K(t) = \Delta_{T(2,n)}(t) = t^{n-1} - t^{n-2} + t^{n-3} - \cdots + +t^2 - t + 1$$

Then either $K = T(2, n)$ or K is hyperbolic.

Proof. Since K is fibered, the Alexander polynomial tells us that K has genus

$$g(K) = g(T(2, n)) = \frac{n - 1}{2}$$

We know that K is either a torus knot, a satellite knot, or hyperbolic ([Thu88]). The only genus-$\frac{n-1}{2}$ torus knots are $T(2, \pm n)$. Thus it suffices to prove that K is not a satellite. Suppose for a contradiction that $K = P(C)$ is a nontrivial satellite knot. By "nontrivial," we mean that the pattern $P \subset S^1 \times D^2$ is not isotopic to the core $S^1 \times 0$, and the companion $C \subset S^3$ is not the unknot. Since K is fibered, the pattern P has winding number $w \geq 1$, and both C and the satellite $P(U)$ are fibered [BZ03, Corollary 4.15, Proposition 5.5]. The Alexander polynomials of these knots are related by

$$\Delta_K(t) = \Delta_P(U)(t) \cdot \Delta_C(t^w),$$

and $\Delta_C(t^w)$ is a nontrivial polynomial with degree $w \cdot g(C) \geq 1$. Since $\Delta_K(t)$ is irreducible, we must then have $\Delta_P(U)(t) = 1$. But since $P(U)$ is also fibered, this can only happen if it has genus zero, meaning that $P(U)$ is the unknot. We now have $\Delta_K(t) = \Delta_C(t^w)$, which then forces $w = 1$. Since P has winding number one and $P(U)$ is the unknot, a result of Hirasawa, Murasugi, and Silver [HMS08, Corollary 1] says that $K = P(C)$ can only be fibered if P is isotopic to the core

$S1 \times \{0\} \subset S1 \times D2$. But this is a contradiction, so K must not be a satellite after all. □

The following result due to Hedden will be needed for the quasipositivity of K:

Theorem 3.3 ([Hed10] Theorem 1.2)**.** *Let K be a fibered knot in S^3. Then $\tau(K) = g_4(K) = g(K)$ if and only if K is strongly quasipositive.*

Since

$$\tau(K) = g_4(K) = g(K) = 3,$$

K must be strongly quasipositive.

Lemma 3.4 ([BHS25] Lemma 3.3)**.** *Let $K \subset S^3$ be a hyperbolic, fibered, strongly quasipositive knot with associated open book (S, h). Then h is freely isotopic to a pseudo-Anosov homeomorphism*

$$\psi : S \to S$$

whose stable foliation has $n \geq 2$ prongs on ∂S, and h has fractional Dehn twist coefficient $c(h) = 1/n$.

Lemma 3.5. *Let $\phi : \Sigma \to \Sigma$ be a pseudo-Anosov map of a closed genus-3 surface Σ. Suppose ψ is symmetric. Then there exists a branched double covering*

$$\pi \colon \Sigma \to S^2,$$

branched along eight points $q_1, ..., q_8 \in S^2$, such that ϕ is a lift of a pseudo-Anosov map

$$b \colon (S^2, q_1, ..., q_8) \to (S^2, q_1, ..., q_6)$$

of the marked sphere, and the invariant foliations of ϕ are lifts of those of b.

Proof. By assumption, ϕ is isotopic to a map ϕ_0 such that

$$\phi_0 \tau = \tau \phi_0$$

where τ is the hyperelliptic involution. Let π be the quotient map under the action of this map. Then

$$\pi \colon \Sigma \to S^2$$

is a branched double covering, branched along eight points $q_1, ..., q_8 \in S^2$. Let $p_1, \dots, p_8$ be the preimages of these branch points,

$$p_i \coloneqq \pi^{-1}(q_i).$$

This commutativity implies that ϕ_0 is in fact a map

$$\phi_0 \colon (\Sigma, p_1, \dots, p_8) \to (\Sigma, p_1, ..., p_8)$$

of the marked genus-2 surface, and is a lift of a map

$$b_0 \colon (S2, q_1, ..., q_8) \to (S^2, q_1, ..., q_8)$$

□

Theorem 3.6. *Let $K \neq T(2, 7)$ be a knot in S^3 such that*

$$\widehat{HFK}(K) \cong \widehat{HFK}(T(2, 7)$$

as bigraded vector spaces, and suppose the fibering monodromy of K is symmetric. Then there exists a pseudo-Anosov 7-braid β whose closure $B = \hat{\beta}$ is an unknot with braid axis A, such that K is the lift of A in the branched double cover $\Sigma(S^3, B) \cong S^3$. In particular, K is a doubly-periodic knot with unknotted quotient A and axis B.

Proof. Suppose K satisfies the hypotheses of the theorem. Then, by Theorem 3.3, K is a genus-2, fibered, strongly quasipositive knot. By Lemma 3.2, K is hyperbolic. By Lemma 3.4, h is then freely isotopic to a pseudo-Anosov map ψ whose invariant foliations have $n \geq 2$ prongs on ∂S. Since the invariant foliations of ψ have more than one boundary prong, ψ extends to a pseudo-Anosov homeomorphism

$$\hat{\psi}\colon \hat{S} \to \hat{S}$$

of the closed genus-2 surface $\hat{S}$ obtained from S by capping off its boundary with a disk, as discussed in §2. The invariant foliations for ψ extend to invariant foliations for $\hat{\psi}$ in which the n boundary prongs extend to an n-pronged singularity (or smooth point if $n = 2$) p in the disk, which is fixed by $\hat{psi}$. It follows from Theorem 3.1 that p is the only fixed point of $\hat{psi}$. By Lemma 3.5, there exists a branched double covering

$$\pi\colon \hat{S} \to S^2$$

of the sphere along eight points $q_1, ..., q_8$ such that $\hat{\psi}$ is the lift of a pseudo-Anosov map

$$b\colon (S^2, q_1, ..., q_8) \to (S^2, q_1, ..., q_6)$$

of the marked sphere. Let

$$\tau\colon \hat{S} \to \hat{S}$$

be the associated covering involution, and note that by assumption $\tau \circ [\hat{\psi}] = \hat{\psi} \circ \tau$. Moreover, the fixed points of τ are precisely the preimages $p_1, ..., p8$, where

$$p_i := \pi^{-1}(q_i)$$

We claim that the fixed point p of $\hat{\psi}$ is one of these p_i; that is,

$$\tau(p) = p \tag{3.1}$$

To see this, we note that

$$\hat{\psi}(\tau(p)) = \tau(\hat{\psi}(p)) = \tau(p).$$

That is, $\tau(p)$ is also a fixed point of $\hat{\psi}$. Since ψ has only one fixed point, 3.1 follows.

Without loss of generality, let us suppose $p = p_8$. Then π restricts to a branched double covering of punctured surfaces

$$\pi' = \hat{S} \setminus \{p_6\} \to S^2 \setminus \{q_8\}$$

We will view these punctured surfaces as the interiors of S and D^2. Let us then extend π' to a branched double covering between compact surfaces,

$$\pi'\colon S \to S^2,$$

branched along the seven marked points $q_1, ..., q_7$. The extension of

$$\hat{\psi}|_{\hat{S}\setminus\{p_8\}} \cong int(S)$$

to S is freely isotopic to h. It follows that h is isotopic to the lift under π' of a homeomorphism

$$\beta\colon (D^2, q_1, \ldots, q_7) \to (D^2, q_1, \ldots, q_7)$$

of the marked disk which is the identity on ∂D^2, where β is freely isotopic to the extension of

$$b|_{S^2\setminus\{q_7\}} \cong int(D^2)$$

to D^2. In what follows, we will think of β as a homeomorphism of this marked disk and as a 7-braid, interchangeably.

This map specifies an open book decomposition (D^2, β) of S^3 with unknotted binding A.

In this open book decomposition, the points $q_1, ..., q_7$ sweep out the closure

$$B = \hat{\beta} \subset S^3$$

of the 7-braid β, with axis A.

The covering map π' extends to a branched double covering from the open book decomposition specified by (S, h) to the open book decomposition specified by (D^2, β), in which the branch set is the braid closure B. Precisely, this extension is defined by

$$\pi' \times id \colon M_h \to M_\beta, \tag{3.2}$$

where $M_h \cong S^3$ is the manifold associated to the open book (S, h), given by

$$M_h \colon (S \times [0,1])/\sim,$$

where $\sim$ is the relation defined by

$$(x, 0) \sim (h(x), 1) \quad \text{for } x \in S$$

$$(x, t) \sim (x, s) \quad \text{for } x \in \partial S \quad \text{and } s, t \in [0,1]$$

and likewise for $M_\beta \cong S^3$. Since $M_h \cong \Sigma(S^3, B) \cong S^3$, it follows that B is an unknot [Wal69]. Finally, the binding

$$K = (\partial S \times \{0\})/\sim$$

of (S, h) is the lift of the binding

$$A = (\partial D^2 \times \{0\})/\sim$$

of (D^2, β) (and braid axis of β) under the branched double covering (3.2). □

4. Exchangeable braids and computations

Theorem 4.1. *Let β be a pseudo-Anosov 7-braid with unknotted closure. Let K be the lift of the braid axis in the branched double cover $\Sigma(S^3, \hat{\beta}) \cong S^3$. If K is strongly quasipositive with Alexander polynomial*

$$\Delta(t) = t^3 - t^2 + t - 1 + t^{-1} - t^{-2} + t^{-3}$$

then β is either not exchangeable or one of the following two braids:

Lemma 4.2 ([BHS25] Lemma 4.2)**.** *Let β be an exchangeable n-braid for some odd n. Suppose that the lift of the braid axis in the branched double cover $\Sigma(S^3, \hat{\beta}) \cong S^3$ is strongly quasipositive. Then β is conjugate to a braid of the form*

$$\beta' = \sigma_{i_1 j_1} \cdot \sigma_{i_2 j_2} \cdot \cdots \cdot \sigma_{i_{n-1} j_{n-1}},$$

where

$$\sigma_{ij} = (\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}) \cdot \sigma_i \cdot (\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}) - 1$$

for $1 \leq i < j \leq n$.

Lemma 4.3 ([BHS25] Lemma 4.3)**.** *If β is an exchangeable n-braid, then the closure of β^k is a fibered link for all integers $k \geq 1$.*

proof of Theorem 4.1. We apply Lemma 4.2 in the case $n = 7$. There are twenty-one generators σ_{ij} with $1 \leq i < j \leq 7$. Up to conjugacy, β is a product of six such generators, according to Lemma 4.2, so there are 85,766,121 braids to check. We examine each of these 85,766,121 braids using Sage [Sag21]. We then can use a built-in functionality of Sage to determine whether any one of these braids are pseudo-Anosov.

Following [BHS25] again, Since K is the lift of the braid axis in the branched double cover $\Sigma(S^3, \hat{\beta})$, its Alexander polynomial $\Delta_K(t)$ is the characteristic polynomial of the reduced Burau representation at $t = -1$, which has the form

$$\rho\colon B_5 \to GL_4(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t \to -1} GL_4(\mathbb{Z})$$

Thus, we can check for the braids β for which for which

$$\Delta_K(t) = t^3 - t^2 + t - 1 + t^{-1} - t^{-2} + t^{-3}$$

by computing for its Burau matrix evaluated at $t = -1$, and see if the characteristic polynomial of this matrix matches this desired one.

Among the 85,766,121 Stallings 7-braids, we found 248,305 braids that satisfy these conditions. Many of these are in fact conjugate to one another, so we then check for duplicate conjugacy classes and only keep one from each class. We found that there are only 24 distinct conjugacy classes of braids among these.

We now apply Lemma 4.3 to these 24 braids, by taking powers of them, computing their Alexander polynomials for the leading coefficient. If the leading coefficient of the Alexander polynomial of a braid power β^k is 1, then we know β^k is fibered, thus by Lemma 4.3 β is an exchangeable 7-braid.

We found that all but two braids from the 24, a power of $k \leq 4$ is sufficient to make β^k fibered. The two exceptional braids are:

$$\begin{aligned}\beta_1 =&\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\\ &\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6\sigma_5\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\end{aligned}$$

and

$$\begin{aligned}\beta_2 =&\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\\ &\sigma_4\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_6\sigma_5\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\end{aligned}$$

For these two braids, we wrote a script to continuously check for fiberedness of β_i^k, and after letting it run for more than 48 hours and checking up to around $k = 10000+$, we terminated the search.

□

5. The proof of Theorem 1.1

proof of Theorem 1.1. Let us assume for a contradiction that $K \neq T(2,7)$. Then Theorem **??** implies that there exists a pseudo-Anosov 7-braid β with unknotted closure B, such that K is the lift of the braid axis A in the branched double cover $\Sigma(S^3, B) \cong S^3$. In particular, K is a doubly-periodic knot with unknotted quotient A and axis B. Below, we prove that A is also braided with respect to B. □

6. Future Works

Appendix A. Computational Details and Code

The following function in Sage outputs all 7-braids σ_{ij} as described in Lemma 4.2, i.e. all elementary generators of Stallings 7-braids with exponent:

```
def elembraids(n=7,positive=True):
  ## Return a tuple of the elementary braids on n strands.
  BG = BraidGroup(n)
  eblist = []
  for i in range(1,n):
    for j in range(i+1,n+1):
      conjugator = BG( list(range(j-1,i,-1)) )
      s = conjugator * BG([i]) * conjugator**(-1) ## exchange strands i,j
      if positive:
        eblist.append(s) ## positive generators only
      else:
        eblist += [s, s**(-1)]
  return tuple(eblist)
```

Listing 1. A function in Sage that outputs all generators of Stallings 7-braids

The following function takes all of the generators that elembraids outputs, generates all of the Stallings 7-braids β' (with exponent 1). It then goes through these braids and only keeps the ones whose closure is the unknot, is pseudo-Anosov, and that for which the characteristic polynomial of the Burau matrix at $t = -1$ is $t^3 - t^2 + t - 1 + t^{-1} - t^{-2} + t^{-3}$.

```
def stallings(n=7,positive=True,verbose=False):
  eblist = elembraids(n,positive)
  answers = []
  itercount = 0
  for braidtuple in cartesian_product_iterator( [eblist]*(n-1) ):
    itercount += 1
    b = prod(braidtuple)
    if b.components_in_closure() == 1: ## it's a knot, so it's an unknot
      A = b.burau_matrix(reduced=True)
      ## chech Alexander poly same as T(2,7) and also braid is pseudoanosov
      if bool(A.subs(t=-1).characteristic_polynomial() == x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) == True and b.is_pseudoanosov() == True:
        answers.append(b)
    if verbose and itercount%1000 == 0:
      print("%d braids checked out of %d, %d successful"%(itercount,len(eblist)**(n-1),len(answers)))
      print("--- %s seconds ---" % (time.time() - start_time))

  return answers
```

Listing 2. A function in Sage that outputs all candidate Stallings 7-braids β

The function `stallings` found 248,305 candidate Stallings 7-braids. Now we will use the following function to further reduce this number:

```
def check_stallings_braids(verbose=True):


  print("Generating the Stallings 7-braids with correct Alexander polynomial and are pseudoanosov...")
  sb_list_all = stallings(7,True,verbose)
  print("%d candidate braids found"%(len(sb_list_all)))
  print("--- %s seconds ---" % (time.time() - start_time))


  # Take one braid representative of each conjugacy class
  sb_list = [BraidGroup(7)([6,5,4,3,2,1])] # start with a braid giving T(2,7)
  itercount = 1
  for b in sb_list_all:
    itercount += 1
    if not True in [b.is_conjugated(beta) for beta in sb_list]:
      sb_list.append(b)
    print("%d candidate braids checked out of %d, %d added to distinct conjugacy classes"%(itercount,len(sb_list_all),len(sb_list)))
  print("%d distinct conjugacy classes found, including %s"%(len(sb_list),str(sb_list[0])))
  print("--- %s seconds ---" % (time.time() - start_time))

  print("Checking that 4th powers of braids are not fibered...")
  still_successful = True
  for sb in sb_list[1:]: # check all the classes that don't lift to T(2,7) because the 0th index is T(2,7)
    ap = (sb**4).alexander_polynomial()
    if abs(ap.coefficients()[0]) == 1:
      print("*** braid might be exchangeable:")
      print("***    b=%s"%str(sb.Tietze()))
      still_successful = False

  if still_successful == False:
    print("--- %s seconds ---" % (time.time() - start_time))
    return False
  else:
    print("All exchangeable braid axes lifted to T(2,7).")
    print("--- %s seconds ---" % (time.time() - start_time))
    return True

print(check_stallings_braids())
```

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