Geometry/Topology II (Differential Geometry)

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Prof. Dr. Jan P, Biringer
Boston College

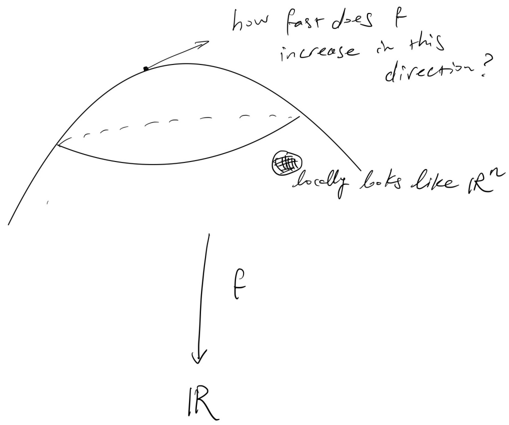
Edited by
Kevin Yeh

Contents

Chapter 1

Manifolds

We are interested in studying spaces that are "locally modelled on $\mathbb{R}^n$", on which one can do calculus. For instance, we may have a function f from such a space to $\mathbb{R}$, and we may ask the rate of change of such a function at a point in some particular direction.



§1.1 Basic constructions

Definition 1.1.1: Topological manifold

A *topological n-manifold* is a Hausdorff, second-countable topological space that is locally Euclidean, i.e. for each point $p \in M$, there exists a neighborhood of p homeomorphic to an open subset of $\mathbb{R}$.

Recall that *Hausdorff* means that for every pair of points $p, q \in M$ there is a pair of disjoint neighborhoods U and V of p and q respectively. Second-countable means that there exists a countable basis $\mathcal{B}$. That is, $\mathcal{B}$ is a countable collection of open sets $\mathcal{B} = \{U_i\}$ such that every open set $U \subset M$ is the union of some of the U_i's.

Example 1.1.2

Trivially, $\mathbb{R}^n$ is itself a manifold. It is clearly Hausdorff; and it can be seen to be second-countable by taking the countable basis to be the balls centered at $\mathbb{Q}^n$ with rational radii.

Both the Hausdorff and the second-countable conditions are preserved under taking subsets; while the locally Euclidean condition is preserved under taking *open subsets*. Thus open subsets of manifolds are manifolds. In particular open subsets of $\mathbb{R}^n$ are manifolds.

There are lots of examples of manifolds in nature that arise as subsets of $\mathbb{R}^n$ cut out by some equations.

Example 1.1.3: Locally Euclid. sp. but not Hausdorff and/or 2nd-count.

from homework.

Theorem 1.1.4: Classification of 1-manifolds

Any connected 1-manifold is homeomorphic to either $\mathbb{R}$ or a circle.

Proof. Omitted. See the work of Gale in the Dropbox. ■

Remark 1.1.5: Equivalent definition, topological manifold

In the definition of a topological manifold, one can equivalently require every point $p \in M$ to have a neighborhood homeomorphic to $\mathbb{R}^n$, or homeomorphic to an open ball in $\mathbb{R}^n$. Indeed, if p has a neighborhood U with $U \cong \hat{U} \subset \mathbb{R}$ (via f), then take a ball $B \subset \hat{U}$ containing $f(p)$. Thus $f^{-1}(B)$ is a neighborhood of p homeomorphic to a ball in $\mathbb{R}^n$ and hence also homeomorphic to $\mathbb{R}^n$. The reverse equivalency is easy.

Definition 1.1.6: Coordinate Chart

If M is an n-manifold, a *coordinate chart* is a map

$$\phi\colon M \supset U \xrightarrow{\cong} \hat{U} \subset \mathbb{R}^n$$

Definition 1.1.7: Local Coordinates

For each p, if $\phi\colon U \xrightarrow{\cong} \hat{U}$ is a coordinate chart about p (i.e. $p \in U$), then we can write

$$\phi(p) = (x^1(p), \ldots, x^n(p)) \in \hat{U} \subset \mathbb{R}^n.$$

We call the $x^i(p)$'s the *local coordinates of p in U*. Sometimes when talking about the point p we may just refer to its local coordinates instead of p.

Example 1.1.8: Empty Set is Manifold of any Dimension

The empty set $\emptyset$ is an n-manifold for every n.

Example 1.1.9: Sphere is manifold

Consider the unit sphere in $\mathbb{R}^{n+1}$:

$$S^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}.$$

There are multiple ways to write down charts for the sphere, here's one way: Let

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \colon x^i > 0\} \subset S^n$$
$$U_i^- = \{(x^1, \ldots, x^{n+1}) \colon x^i < 0\} \subset S^n$$

First of all, these are all open subsets of S^n and they cover S^n. We then define the maps

$$\phi_i^\pm \colon U_i^\pm \to B_1(0) \subset \mathbb{R}^n$$
$$(x^1, \ldots, x^{n+1}) \mapsto (x^1, \ldots, \hat{x^i}, \ldots, x^{n+1})$$

which is a homeomorphism with inverse

$$(x^1, \ldots, x^n) \mapsto \left(x^1, \ldots, x^{i-1}, \sqrt{1 - \sum_i^n x_i^2}, x^i, \ldots, x^n\right).$$

The Hausdorff and second-countable conditions are trivial since S^n is a subset of $\mathbb{R}^{n+1}$.

Example 1.1.10: Graph of continuous map is manifold

Suppose $U \subset \mathbb{R}^n$ is open, and $f \colon U \to \mathbb{R}^k$ is a continuous map. Define the *graph* of f to be

$$\Gamma(f) = \{(x, f(x)) \colon x \in U\} \subset \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}.$$

Consider the projection onto the first factor:

$$\phi \colon \Gamma(f) \to U \subset \mathbb{R}^n$$

which is a homeomorphism with inverse

$$x \mapsto (x, f(x)).$$

This makes $\Gamma(f)$ an n-manifold (with a single chart), embedded as a subset of $\mathbb{R}^{n+k}$.

The fact that graphs of continuous maps are manifolds also allow us to see that the sphere is a manifold since it is “locally a graph".

Proposition 1.1.11: Product of Manifolds is Manifold

Suppose M and N are m- and n-manifolds respectively. Then $M \times N$ is an $m+n$ manifold

Proof. Given charts

$$\phi\colon M \supset U \to \hat{U} \subset \mathbb{R}^m$$

and

$$\psi\colon N \supset V \to \hat{V} \subset \mathbb{R}^n$$

we can define

$$\phi \times \psi\colon M \times N \supset U \times V \to \hat{U} \times \hat{V} \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

by

$$\phi \times \psi(U, V) = (\phi(U), \psi(V)).$$

Then $\phi \times \psi$ is a chart for $M \times N$, and any (U, V) is in the domain of such a chart. ■

The above proposition immediately implies that the n-torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$$

is an n-manifold.

Example 1.1.12: Real projective space is manifold

Consider real projective n-space, defined as

$$\mathbb{R}P^n = S^n / x \sim -x$$

with the quotient topology. Recall that the quotient topology is the following: if $\pi\colon S^n \to \mathbb{R}P^n$ is the projection, then $U \subset \mathbb{R}P^n$ is open if and only if $\pi^{-1}(U) \subset S^n$ is open.

Recall from Example 9 that there are charts on S^n

$$\phi_i^+ : U_i^+ \to \mathbb{R}^n$$

where

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \in S^n \colon x^i > 0\}.$$

We claim that for all i, $\pi\big|_{U_i^+}$ is a homeomorphism onto its image $V_i \subset \mathbb{R}P^n$. Indeed, $\pi\big|_{U_i^+}$ is injective since no pair of antipodal points on the sphere is contained in the same hemisphere, it is surjective since π is a projection; it is continuous simply by being a projection; and it is an open map since if $W \subset U_i^+$ is open, then

$$\pi^{-1}(\pi(W)) = W \cup -W$$

which is open in S^n, thus $\pi(W)$ is open in the quotient topology. Combining these three properties of $\pi\big|_{U_i^+}$ gives us the homeomorphism. Therefore, we have the following charts for $\mathbb{R}P^n$:

$$\phi_i^+ \circ \left(\pi\big|_{U_i^+}\right)^{-1} : V_i \to B_1(0) \subset \mathbb{R}^n.$$

§1.2 Covering Spaces and Group Actions

Proposition 1.2.1

Suppose $f\colon X \to Y$ is a covering map and X is second-countable and Y is Hausdorff?. Then X is an n-manifold if and only if Y is.

Firstly, the requirement that X be second-countable is necessary, otherwise we may have something like this: Let $\mathbb{R}$ be equipped with the discrete topology, so it is not second-countable and thus not a manifold. Then

$$\mathbb{R} \to \{0\}$$

is (trivially) a covering map but $\{0\}$ is a 0-manifold.

Proof. We omit the proof for the Hausdorff and second-countable statements.

Suppose X is locally Euclidean. Given $y \in Y$, find a neighborhood U of y that is evenly covered:

$$f^{-1}(U) = \coprod_i U_i$$

where each $f\big|_{U_i}\colon U_i \to U$ is a homeomorphism. Pick some i, and let $x = f\big|_{U_i}(y)^{-1}$, and pick a neighborhood V of x with $V \cong \mathbb{R}^n$ (from X being locally Euclidean). Without loss of generality we can assume $V = U_i$. Then $f(V)$ is a neighborhood of y that is homeomorphic to $\mathbb{R}^n$, showing that Y is locally Euclidean. The other direction is similar. ■

Motivated by the above proposition, we would like to study covering spaces of manifolds. We outline some facts relating covering spaces with group actions.

Definition 1.2.2: Properly discontinuous action

Suppose that Γ is a group acting by homeomorphisms on a manifold X, i.e. we have a group homomorphism

$$\Gamma \to \text{Homeo}(X).$$

We say that the action is *properly discontinuous* if for all compact subset $K \subset X$, we have that

$$\{\gamma \in \Gamma \colon \gamma(K) \cap K \neq \emptyset\}$$

is a finite set.

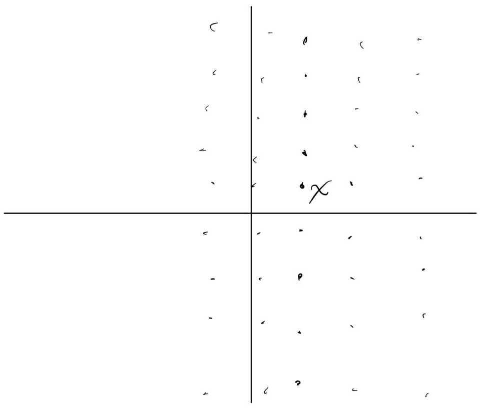
Example 1.2.3

1. If X is compact, then the action of Γ on X is properly discontinuous if and only if Γ is a finite group.
2. $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ via $v(x) = x + v$ for $v \in \mathbb{Z}^n$ and $x \in r^n$. Thus $v \in \mathbb{Z}^n$ corresponds to the homeomorphism of $\mathbb{R}^n$ that is translation of v. Now if $K \subset \mathbb{R}^n$ is compact, then $K \subset B_R(0) \subset \mathbb{R}^n$ for some radius R. If

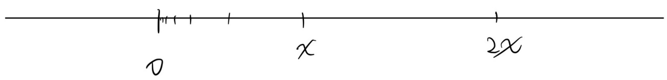
   $$v(K) \cap K \neq \emptyset$$

   for some $v \in \mathbb{Z}^n$, it implies that there exists $x \in \mathbb{R}^n$ such that $|x| \leq R$ and $|x + v| \leq R$, which together with the triangle inequality, means that

$|v| \leq 2R$. But there exists only finitely many $v \in \mathbb{Z}^n$ with norm less than or equal to $2R$, thus the action is properly discontinuous. It is often useful to draw the orbit of an element to visualize a group action. In this case, the orbit of an element consists of integer translates of it:



3. If Γ is infinite and has a global fixed point, i.e. there exists $x \in X$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$, then Γ does not act properly discontinuously. This can be seen by taking $K = \{x\}$. For example, take the action of $\mathbb{Z}$ on $\mathbb{R}$ via $n(x) = 2^n x$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Here the action fixes $0 \in \mathbb{R}$. here we have orbits that look like this:



Definition 1.2.4: Free action

A group action of a group Γ acting on a set X is *free* if there exists no $\gamma \in \Gamma \setminus e_\Gamma$ that fixes a point of X.

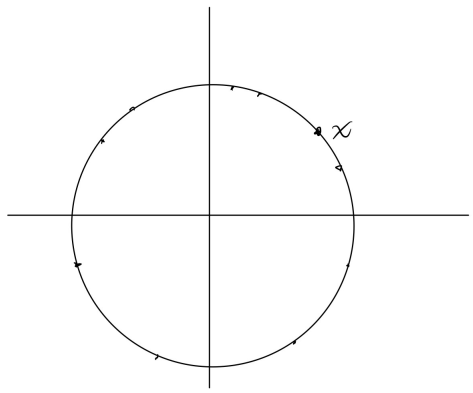
Example 1.2.5

1. The action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ in the previous example is free.
2. An example of a properly discontinuous but non-free action is the following: Consider $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{R}^2$ where the nontrivial element acts via $(x, y) \mapsto (x, -y)$.
3. An example of a free but not properly discontinuous action is the following:

Consider $\mathbb{Z}$ acting on $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\} \subset \mathbb{C}$ via

$$n(e^{i\theta}) = e^{i(\theta+n\alpha)}$$

where we fix some real number α such that $\frac{\alpha}{\pi}$ is irrational. The the orbits look like



where we never come back to any of the same points as the action rotates each point. More precisely,

$$\theta + n\alpha = \theta \mod 2\pi$$

if and only if

$$n = 0.$$

Thus the action is free. The action is not properly discontinuous because S^1 is compact and $\mathbb{Z}$ is not finite.

Theorem 1.2.6

If X is a locally compact Hausdorff space and Γ acts on X freely and properly discontinuously by homeomorphisms, then the projection

$$\pi\colon X \to \Gamma \backslash X$$

is a covering map.

Here $\Gamma\backslash X$ is the set of all orbits Γ_x equipped with the quotient topology given by $\pi(x) = \Gamma_x$.

Recall that *locally compact* means that for any $p \in X$, there exists a neighborhood of X with compact closure. For instance, all manifolds are locally compact, since under a chart $\phi\colon U \to \hat{U}$, we can take a small enough ball around $\phi(x)$ inside $\hat{U}$, and the preimage of this ball is the desired neighborhood around x with compact closure.

Corollary 1.2.7

If X is a manifold and Γ acts on X freely and properly discontinuously, then $\Gamma\backslash X$ is a manifold.

Proof of Theorem 6. Given $x \in X$, pick a neighborhood U of x with compact closure. Then the set

$$\Delta = \{\gamma \colon \gamma(\overline{U}) \cap \overline{U} \neq \emptyset\}$$

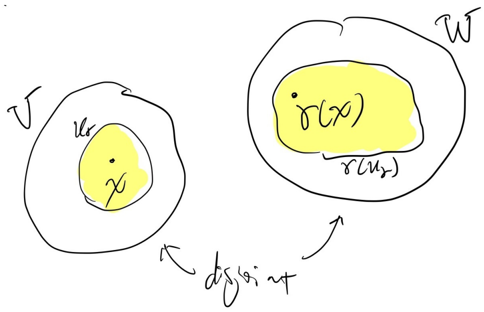
is finite by definition of properly discontinuous action. Now by freeness of the action, no $\gamma \in \Gamma$ fixes x. Thus for any $\gamma \in \Gamma \setminus e_\Gamma$ there exist disjoint open neighborhoods V and W, of x and $\gamma(x)$ respectively (by Hausdorffness). Since the action is by homeomorphisms, so in particular the action is continuous, $\gamma^{-1}(W)$ is open. Then set

$$U_\gamma = V \cap \gamma^{-1}(W)$$

which is an open neighborhood of x contained inside V with the property that $\gamma(U_\gamma) \subset W$ therefore

$$U_\gamma \cap \gamma(U_\gamma) = \emptyset.$$

Here's a diagram of what all this looks like up to this point:



Now set

$$O = \bigcap_{\gamma \in \Delta} U_\gamma \cap U.$$

Then O is a neighborhood of x since Δ is finite so O is a finite intersection of neighborhoods of x. Further, O has the property that

$$\gamma(O) \cap O = \emptyset$$

for all $\gamma \in \Gamma \setminus e_\Gamma$ non-trivial elements of Γ. This is because if $\gamma \notin \Delta$ then $\gamma(U) \cap U = \emptyset$ by the definition of Γ; on the other hand if $\gamma \in \Delta$, then $\gamma(U_\gamma) \cap U_\gamma = \emptyset$ (?????)

It follows that the sets

$$\gamma O, \gamma \in \Gamma$$

are all disjoint. Indeed if

$$\alpha O \cap \beta \neq \emptyset$$

then

$$O \cap \alpha^{-1}\beta O \neq \emptyset$$

contradicting 1.2. But then if we set

$$V = \{\Gamma_x \colon x \in O\}$$

we have

$$\pi^{-1}(V) = \bigsqcup_{\gamma \in \Gamma} \gamma O$$

and π restricts to a homeomorphism

$$\pi\big|_{\gamma O} \colon \gamma O \to V$$

for each γ. ■

Example 1.2.8

1. Consider
$$\mathbb{R}P^n = \mathbb{Z}/2\mathbb{Z}\backslash S^n$$
where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts via $x \mapsto -x$. This action is free and properly discontinuous, so we get another proof that $\mathbb{R}P^n$ is a manifold.

2. If $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ via $v(x) = x + v$ for $v \in \mathbb{Z}^n, x \in \mathbb{R}^n$, then $\mathbb{Z}^n\backslash\mathbb{R}^n$ is an n-manifold. In fact,
$$\mathbb{Z}^n\backslash\mathbb{R}^n \cong S^1 \times \cdots \times S^1 =: T^n$$
(verify this).

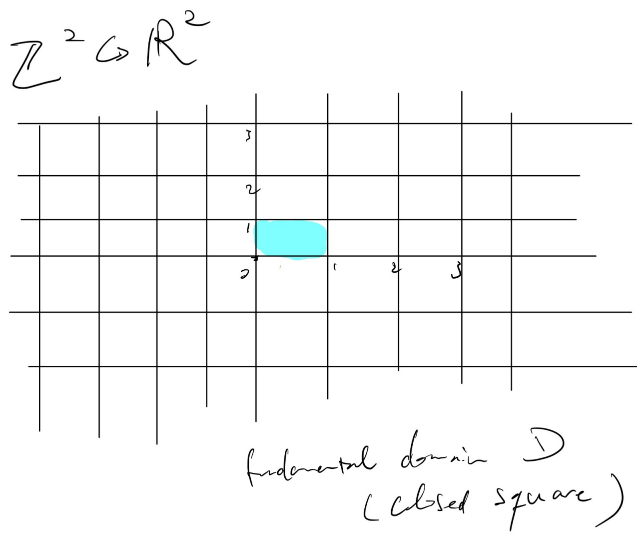
Definition 1.2.9: Fundamental domain

If Γ acts on X, a *fundamental domain* for the action is a closed set $D \subset X$ such that

1. $\text{Int}(D) \cap \gamma(\text{Int}(D)) = \emptyset$.
2. $\bigcup_\gamma \gamma(D) = X$.

Example 1.2.10: Fundamental domain of $\mathbb{Z}^2$ acting on $\mathbb{R}^2$

A fundamental domain is the following closed square:



Example 1.2.11: Fundamental domain of $\mathbb{Z}/2\mathbb{Z}$ acting on S^2

We can take any closed hemisphere.

Because of the second condition in the definition of fundamental domain,

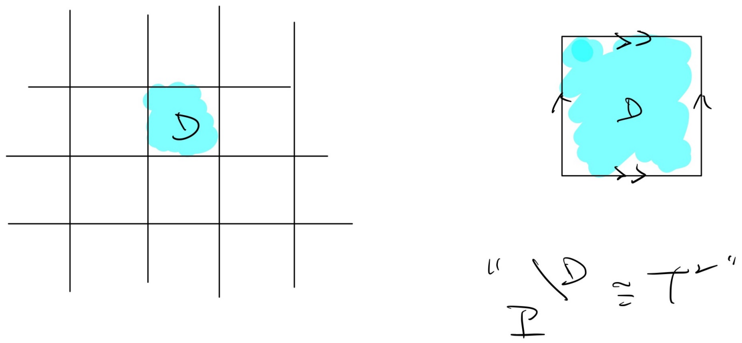
$$\pi \colon X \to {}_{\Gamma}\backslash^{X}$$

restricts to a surjection on D, and hence to a quotient map on D, so

$${}_{\Gamma}\backslash^{X} \cong \text{“}{}_{\Gamma}\backslash^{D}\text{”}$$

where the RHS is the quotient of D by the equivalence relation $x \sim y$ if there exists $\gamma \in \Gamma$ such that $\gamma(x) = y$.

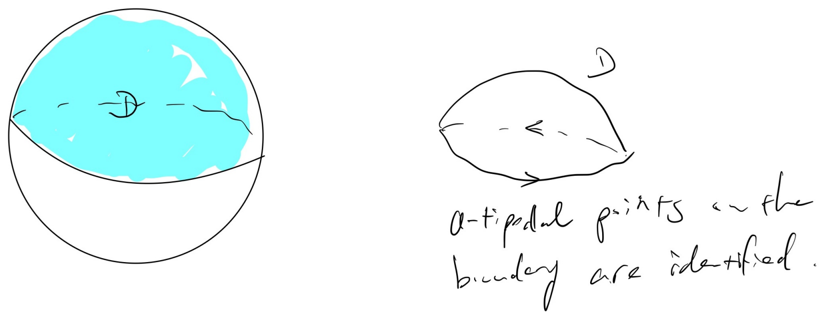
Fundamental domains are helpful since only boundary points of D are identified, and one can often take D to be a polygon where sides are identified by the Γ-action. For instance, for $\mathbb{Z}^2$ acting on $\mathbb{Z}^2$ we have



giving us

$$\text{“}{}_{\Gamma}\backslash^{D}\text{”} \cong T^2.$$

And for $\mathbb{Z}/2\mathbb{Z}$ acting on S^2 we have



giving us

$$\text{“}{}_{\Gamma}\backslash^{D}\text{”} \cong \mathbb{R}P^2.$$

Announcement: Office hours Tuesdays 2pm, Thursdays 8am.

§1.3 Smooth Manifolds

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^k at $p \in \mathbb{R}^n$ if all the partial derivatives of the component functions f^i of f exist up to order k in a neighborhood of p and are continuous there. So, we require continuity of the functions

$$\frac{\partial^\ell f^i}{\partial x^{j_1} \dots x^{j_\ell}}, \quad i = 1, \dots, m, \ell \leq k$$

in a neighborhood of p. We say f is *smooth*, or C^∞ at p if it is C^k at p for all k. Common examples of smooth functions: exponentials, trig functions, polynomials.

Problem 1. *If M is an n-manifold and $f\colon M \to \mathbb{R}$, is a function, what should it mean for f to be smooth at $p \in M$?*

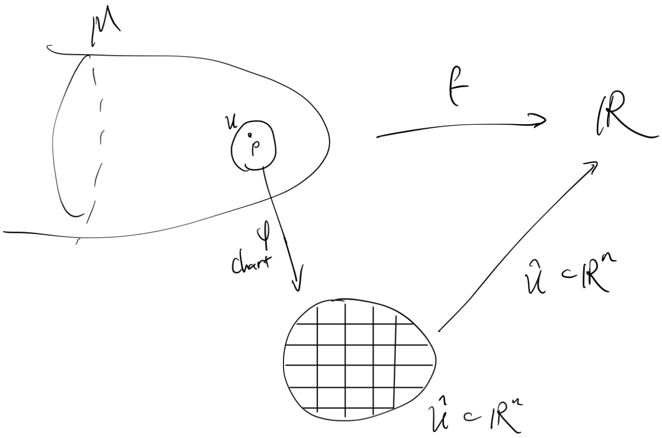


Figure 1.1: We want to say that f is smooth at p if $f \circ \phi(p)^{-1}$ is smooth at $\phi(p)$.

Problem: What if we use a different chart? Say we have two charts around p, $\phi\colon U \to \hat{U} \subset \mathbb{R}^n$ and $\psi\colon V \to \hat{V} \subset \mathbb{R}^n$.

$f \circ \phi^{-1}$ may be smooth at $\phi(p)$, but maybe $f \circ \psi^{-1}$ is not at $\psi(p)$. Thus we need the charts to be "compatible":

If $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are smooth then

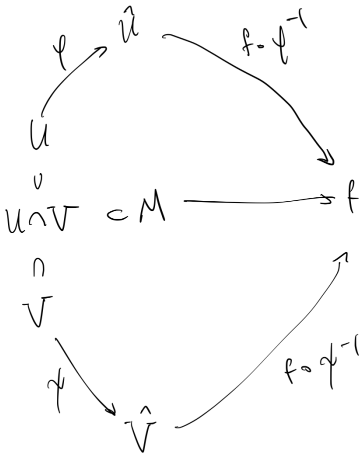
$$f \circ \psi^{-1} = f \circ \phi^{-1} \circ (\phi \circ \psi^{-1}),$$

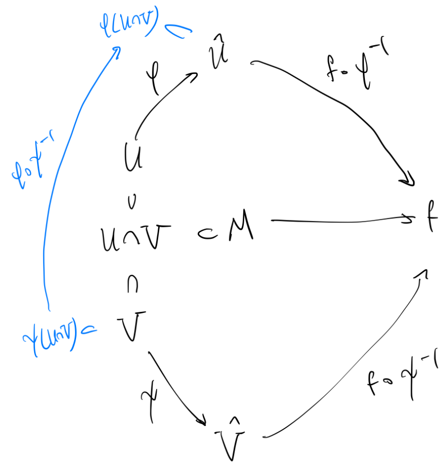
so $f \circ \psi^{-1}$ is smooth at $\psi(p)$ if and only if $f \circ \phi^{-1}$ is smooth at $\phi(p)$ since compositions of smooth functions are smooth.

Definition 1.3.1

If $\phi\colon U \to \hat{U} \subset \mathbb{R}^n$ and $\psi\colon V \to \hat{V} \subset \mathbb{R}^n$ are charts, then

$$\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V)$$





and

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$$

are called the *transition maps* between the two charts. We say that ϕ and ψ are *smoothly compatible* if their transition maps are smooth.

A set of charts whose domains cover m is an *atlas* for M. An atlas $\mathcal{A}$ is called smooth if all its charts are smoothly compatible.

Definition 1.3.2: Smooth function

If M is equipped with a smooth atlas, then $f \colon M \to \mathbb{R}$ is *smooth at* $p \in M$ if

1. there exists a chart (U, ϕ) around p such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$, or equivalently
2. for all charts (U, ϕ) around p, $f \circ \phi^{-1}$ is smooth at $\phi(p)$.

Example 1.3.3

Some smooth atlases function equivalently: on the 1-manifold $\mathbb{R}$ we have atlases

$$\mathcal{A} = \{(\mathbb{R}, \mathrm{Id})\}$$

and

$$\mathcal{B} = \{((x-1, x+1), \mathrm{Id}) : x \in \mathbb{R}\}.$$

But $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-smooth if and only if f is $\mathcal{B}$-smooth if and only if f is smooth in the usual sense.

Example 1.3.4

Still on $\mathbb{R}$, take again

$$\mathcal{A} = \{(\mathbb{R}, \mathrm{Id})\}$$

and

$$\mathcal{B} = \{(\mathbb{R}, x \mapsto x^3)\}.$$

But now the function $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-smooth, yet it is not $\mathcal{B}$-smooth, since

$$\mathrm{Id} \circ (x \mapsto x^3)^{-1} = x \mapsto \sqrt[3]{x}$$

which is not smooth at 0.

Definition 1.3.5: Maximal atlas and smooth structure

A smooth atlas $\mathcal{A}$ is *maximal* if it is not contained in a larger smooth atlas. Equivalently, $\mathcal{A}$ is *maximal* if any chart that is compatible with all charts in $\mathcal{A}$ is in $\mathcal{A}$.

Definition 1.3.6: Smooth structure, smooth manifold

A *smooth structure* on M is a maximal smooth atlas. A *smooth manifold* is a manifold equipped with a smooth structure.

Example 1.3.7

1. $\mathbb{R}^n$ equipped with the “idenity chart" $\{\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n\}$, which generate the maximal atlas

   $$\mathcal{A} = \{\phi : U \to \hat{U} \text{ smooth homeo btw open subsets of } \mathbb{R}^n \text{ with smooth inverse}\}$$

2. If V is an n-dim vetor space, then we can consider the atlas

   $$\mathcal{A} = \{\text{linear isomorphism } L : V \to \mathbb{R}^n\}.$$

This is a smooth atlas, since the transition maps are of the form

$$L' \circ L^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$$

where $L, L' \in \mathcal{A}$, so is smooth (The topology on V is either given by requiring that every such L is a homeomorphism. or alternatively pick a norm $|\cdot|$ on V and set $d(v,w) = |v-w|$ to get a metric on V).

3. If V, W are m, n-dimensional vector spaces, then the space of linear maps $L(V,W)$ is a vector space of dimension mn, and hence is a smooth manifold.

4. We shall consider all 0-manifolds to be smooth manifolds. Here, a 0-manifold is a countable set equipped with the discrete topology, since $\mathbb{R}^0$ is a point. Charts have the form
$$\{pt\} \to \mathbb{R}^0$$
and transition maps $\mathbb{R}^0 \to \mathbb{R}^0$ are just the unique map, which we will consider to be smooth.

5. S^n with the charts which we constructed before form a smooth atlas.

   PIC

$$\begin{aligned}
&\phi_j^+ \circ (\phi_i^+)^{-1}(x^1, \ldots, x^n) \\
&\qquad = \phi_j^+ \left(x^1, \ldots, x^{i-1}, \sqrt{1 - \sum_k (x^k)^2}, x^i, \ldots, x^n \right) \\
&\qquad = \left(x^1, \ldots, x^{i-1}, \sqrt{1 - \sum_k (x^k)^2}, x^i, \ldots, \hat{x^j}, \ldots, x^n \right)
\end{aligned}$$

6. Products of smooth manifolds have a natural smooth structure.

7. Open subset $U \subset M$ where M is a smooth manifold. Since for every chart $\phi \colon V \to \hat{V}$ for M, you get a chart $\phi|_{U \cap V}$ for U. Transition maps are restrictions of transition maps. For example, if V is a vector space, consider

$$\mathrm{GL}(V) = \{\text{linear isomorphisms } V \to V\}$$

which is an open subset of the vector space $L(V,V)$, and hence is a smooth manifold of dimension $(\dim V)^2$. By the way, it is an open subset because

$$\mathrm{GL}(V) = \det{}^{-1}(\mathbb{R} \setminus 0)$$

where $\mathbb{R} \setminus 0$ is open, and $\det \colon L(V,V) \to \mathbb{R}$ is continuous.

Smooth manifold with boundary Smooth manifolds with boundary are defined in the same way, requiring that transition maps between charts

$$\phi \colon U \to \hat{U} \subset H^n$$

are smooth. Note that the transition maps go from open subsets of H^n to open subsets of H^n. Here, if $A \subset \mathbb{R}^n$ a map $f \colon A \to \mathbb{R}^k$ is *smooth* if f extends to a smooth map defined on a neighborhood of A.

Proposition 1.3.8

Every smooth atlas for M is contained in a unique maximal one. Two smooth atlases determine the same maximal one if and only if the charts in one are compatible with the charts in the other.

Proof. ■

Lemma 1.3.9

If M is a set and $\{U_\alpha\}$ are subsets of M with bijections

$$\phi_\alpha \colon U_\alpha \to \hat{U}_\alpha \subset \text{ (open) } \mathbb{R}^n$$

or H^n if you want a manifold with boundary. Such that

1. for all α, β the sets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$ and the transition map $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth.
2. Countably many of the U_α cover M.
3. If $p \neq q$ in M, either there exist $U_\alpha \ni p, q$ or there exist disjoint U_α, U_β containing p, q respectively.

Then there exists a unique smooth structure on M where the ϕ_α are charts.

Proof. See Lee. ■

Here, the topology on M is generated (as a basis) by the preimages $\phi_\alpha^{-1}(V)$, where $V \subset \hat{U}_\alpha$ is open.

You can use this to define a smooth structure on a vector space.

Example 1.3.10

et V be an n-dim vector space. Define

$$G_k(V) = \{k\text{-dim subspace } H \subset V\}$$

this is called the *Grassmannian* of k-dim subspaces of V. We want to show that $G_k(V)$ is naturally a $k(n-k)$-manifold. See Lee for the details of the idea. Given a decomposition $V = A \oplus B$ where $\dim(A) = k$ and $\dim(B) = n - k$, then for any linear map $f \in L(A, B)$, consider the graph of f:

$$\Gamma(f) = \{a + f(a) \colon a \in A\}$$

is a k-dim subspace of V. So we can use

$$\begin{aligned} L(A, B) &\to G_k(V) \\ f &\mapsto \Gamma(f) \end{aligned}$$

as the inverse of a chart for $G_k(V)$. See Lee for details about why transition maps are smooth (they'll turn into matrix additions and multiplications after choosing suitable coordinates).

Definition 1.3.11

If M, N are smooth manifolds, we say that a function $f\colon M \to N$ is *smooth at* $p \in M$ is there exists

PIC

such that $\psi \circ f \circ \phi^{-1}$ is smooth at $\phi(p)$. We call

$$\psi \circ f \circ \phi^{-1}$$

the *the coordinate representation of* f.

We say that f is *smooth* if it is smooth at every $p \in M$.

Example 1.3.12

1. Smooth maps of Euclidean spaces, with repesct to the standard smooth structure on $\mathbb{R}^n$.
2. Constant maps, identity maps.
3. The inclusion $i\colon U \to M$ of an open submanifold. If $p \in U$, take an M-chart $\psi\colon V \to \hat{V}$ around $p = i(p)$. Then

$$\psi\big|_{V\cap U}\colon V \cap U \to \phi(V \cap U)$$

is a chart for U around p. So the coordinate representation is

$$\psi \circ i \circ (\psi\big|_{V\cap U})^{-1} = \mathrm{Id}$$

which is smooth.

4. Consider $A\colon S^n \to S^n$ defined by $A(x) = -x$. This is called the *antipodal map*. If for instance $p \in U_i^+$ then on U_i^+ we have

$$\phi_i^-(-x) = -\phi_i^+(x)$$

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Thus the coordinate representation of our map is $z \mapsto -z$, which is smooth.

Some facts about smooth maps:

1. Smooth maps are continuous.
2. Composition of smooth maps are smooth.

Theorem 1.3.13: Diffeomorphism

A smooth bijection $f\colon M \to N$ with smooth inverse f^{-1} is called a *diffeomorphism*. If there exists a diffeomorphism $f\colon M \to N$ we say M and N are *diffeomorphic*

Example 1.3.14

1. If $S^n_r\{x \in \mathbb{R}^{n+r} : |x| = r\}$, then S^n_r is naturally a smooth manifold (just like with S^n), and if $r, s > 0$,

$$\begin{aligned} S^n_r &\to S^n_s \\ x &\mapsto \frac{s}{r} \cdot x \end{aligned}$$

   is a diffeomorphism. Using the orthogonal projection charts, the coordinate representation of the above will be also $x \mapsto \frac{s}{r} \cdot x$. Thus spheres of different radii are all diffeomorphic.

2. Consider the smooth 1-manifolds

$$(\mathbb{R}, \{\text{ß}\colon \mathbb{R} \to \mathbb{R}\}), (\mathbb{R}, \{x \mapsto x^3\}).$$

   We have the diffeomorphism

$$\begin{aligned} f\colon (\mathbb{R}, \{\text{ß}\colon \mathbb{R} \to \mathbb{R}\}) &\to (\mathbb{R}, \{x \mapsto x^3\}) \\ x &\mapsto \sqrt[3]{x} \end{aligned}$$

   This is a bijection, and in coordinates it is

   PIC

   so the coordinate representation is the identity map, so is smooth. Similarly f^{-1} is smooth.

Remark 1.3.15

Milnor-Kervaire (1963) showed that there exists 15 smooth structures on S^7 up to diffeomorphism. Donaldson-Freedman (1984) showed there exists uncountably many smooth structures on $\mathbb{R}^4$ up to diffeomorphism. In dimensions 1,2,3 any topological manifold admits a unique smooth structure up to diffeomorphism (Rado, Bing, Moire).

Definition 1.3.16: Smooth covering map

If M, N are smooth manifolds, a *smooth covering map* is a map $\pi\colon M \to N$ such that for all $p \in N$ there exists a neighborhood $V \ni p$ such that

$$\pi^{-1}(V) = \bigsqcup_i V_i$$

where each

$$\pi\big|_{V_i} : V_i \to V$$

is a diffeomorphism.

Example 1.3.17

$$\pi : \mathbb{R} \to S^1$$
$$t \mapsto (\cos t, \sin t)$$

is a smooth covering map. If

$$U = \{(x, y) \in S^1 : y > 0\}$$

then

$$\pi^{-1}(U) = \bigsqcup_{k\in\mathbb{Z}} (2\pi k, 2\pi k + \pi).$$

The map

$$U \to (-1, 1)$$
$$(x, y) \mapsto x$$

is a chart for S^1, so in local coordinates π is the map $t \mapsto \cos t$, which is smooth. One can check that the inverse $U \to [2\pi k, 2\pi k + \pi]$ is smooth for all k as well.

Consider

$$\mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$

which is smooth, and is a covering map since it is a homeomorphism, but it is not a smooth covering map.

Proposition 1.3.18

If $X \to Y$ is a (topological) covering map, and Y is a smooth manifold, then there exists a unique smooth structure on X such that π is a smooth covering map

Proof. Idea: Given $x \in X$, pick an evenly covered neighborhood of $\pi(x)$, i.e.

$$\pi(x) \in U, \quad \pi^{-1}(U) = \bigsqcup_i U_i.$$

Shrinking U, we can assume we have a chart $\phi : U \to \hat{U} \subset \mathbb{R}^n$. If $x \in U_i$ then $\phi\circ\pi : U_i \to \hat{U}$ we can take as a chart around x. These form a smooth atlas for X. For local coordinates around $x, \pi(x)$, we can take the charts $\phi \circ \pi$ and ϕ and then the coordinate representation is

$$\pi \circ (\phi \circ \pi)^{-1} = \mathrm{Id}$$

so π is a diffeomorphism $U_i \to U$. ■

Proposition 1.3.19

Suppose X is a smooth manifold and Γ acts on X properly discontinuously and freely by diffeomorphisms. Then there exists a unique smooth structure on $\Gamma\backslash X$ such that the quotient map

$$\pi\colon X \to \Gamma\backslash X$$

is a smooth covering map.

Proof. See Prop. 4.40 in Lee. Idea: Given $p \in X$, let $U \ni p$ be a neighborhood all of whose translates $\gamma(U)$, $\gamma \in \Gamma$, are disjoint. Shrinking U, we can assume it is the domain of a chart $\phi\colon U \to \hat{U} \subset \mathbb{R}^n$. Then $\pi(U) \subset \Gamma\backslash X$ is open and

$$\pi\big|_U\colon U \to \pi(U)$$

is a homeomorphism, so we can take

$$\phi \circ \pi\big|_U^{-1}\colon \pi(U) \to \hat{U}$$

as a chart for $\Gamma\backslash X$ around $\pi(p)$.

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If $\pi(p) = \pi(q) \in \Gamma\backslash X$, then $q = \gamma(p)$ for some $\gamma \in \Gamma$.

PIC

and on $V \cap \gamma(U)$ we have

$$\left(\pi\big|_U\right)^{-1} \circ \pi\big|_V = \gamma^{-1}$$

so near $\psi(q)$, the transition map is $\phi \circ \gamma^{-1} \circ \psi^{-1}$, which is smooth since $\gamma^{-1}\colon X \to X$ is smooth. ■

§1.4 Constructing Smooth Maps

Lemma 1.4.1

The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth.

Proof. PIC

Idea: The point is to show that f is smooth at $t = 0$. Every time you take a derivative, a $-\frac{1}{t^2}$ comes down from an exponent. But for all k,

$$\frac{1}{t^{2k}} \cdot e^{-\frac{1}{t}} \to 0 \quad \text{as } t \to 0$$

because exponentials grow faster than polynomials. This “implies" that $f^{(k)} = 0$ for all k.

See Lee for details. ■

Lemma 1.4.2

Given $0 < r_1 < r_2$, there exists a smooth function $H\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$H \equiv 1 \text{ on } \overline{B_{r_1}(0)}$$

$$0 < H < 1 \text{ on } B_{r_2}(0) \setminus \overline{B_{r_1}(0)}$$

$$H \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{r_2}(0)$$

We call H a *bump function.*

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Proof. Set

$$H(x) = \frac{f(r_2 - |x|)}{f(r_2 - |x|) + f(|x| - r_1)}$$

where f is the function defined in the previous lemma. Then since $0 \leq f \leq 1$, we must also have $0 \leq H \leq 1$.

We have

$$|x| < r_1 \Leftrightarrow f(|x| - r_1) = 0 \Leftrightarrow H = 1.$$

And similarly

$$|x| < r_2 \Leftrightarrow f(r_2 - |x|) = 0 \Leftrightarrow H = 0.$$

It is smooth since f is smooth and the denominator is never 0; and the fact that $|\cdot|$ is not smooth at $x = 0$ does not matter since $H \equiv 1$ in a neighborhood of the origin. ■

Definition 1.4.3

If $f\colon M \to \mathbb{R}$ is a function, we define the *support* of f to be

$$\operatorname{supp}(f) := \overline{\{p \in M \colon f(p) \neq 0\}}.$$

We say f is *compactly supported* if $\operatorname{supp}(f)$ is compact.

For instance, H as defined above has support

$$\operatorname{supp}(H) = \overline{B_{r_2}(0)}$$

and so H is compactly supported.

Example 1.4.4

If M is a manifold and

$$\phi\colon U \to B_3(0) \subset \mathbb{R}^n$$

is a chart, then setting $r_1 = 1$, $r_2 = 2$ above in the construction of H, we can define

$$F\colon M \to \mathbb{R}$$
$$x \mapsto \begin{cases} H \circ \phi(x) & x \in U \\ 0 & \text{otherwise} \end{cases}$$

Then this is a non-constant smooth function on M, which we can also call a bump function.

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If $x \in U$, then we can use ϕ as our chart around x, and the local coordinate representation of F is H, which is smooth. If $x \notin U$, then since $\text{supp}(F)$ is a compact subset of U, so there exists a neighborhood of x on which $F \equiv 0$, so F is smooth at x.

Now we can create many smooth functions on M by summing up bump functions.

Definition 1.4.5: Locally Finite Collection of Subsets

A collection A of subsets of M is *locally finite* if every $p \in M$ has a neighborhood U that intersects only finitely many elements of A.

Definition 1.4.6: Refinement of an Open Cover

If A is an open cover of M, a *refinement* of $\mathcal{A}$ is another open cover $\mathcal{R}$ such that for any $R \in \mathcal{R}$ there exists $A \in \mathcal{A}$ with $R \subset A$.

Definition 1.4.7: Paracompact

We say M is *paracompact* if every open cover of M admits a locally finite refinement.

Example 1.4.8

An open cover of $\mathbb{R}$ that is not locally finite:

$$\mathcal{O} = \{(a, b) \colon a < 0, b > 0\}$$

This is not locally finite since there exists infinitely many intervals and they all contain 0.

A locally finite refinement of $\mathcal{O}$ is

$$U = \{(x-2, x+2) \colon x \in \mathbb{Z}\}$$

since given $p \in \mathbb{R}$, only finitely many intersect $(p-1, p+1)$. Also, for all x,

$$(x-2, x+2) \subset (-|x-2|-1, ||)$$

Theorem 1.4.9: Existence of Locally Finite Refinement for a Cover on Manifold

Let M be a topological manifold and $\mathcal{B}$ be a basis of open sets. If $\mathcal{A}$ is an open cover of M, then there exists a locally finite refinement $\mathcal{R}$ of $\mathcal{A}$ with $\mathcal{R} \subset \mathcal{B}$. In particular, manifolds are paracompact.

Lemma 1.4.10

M is σ-compact, i.e. there exists a sequence $K_1 \subset K_2 \subset \dots$ of compact sets with

$$\bigcup_i K_i = M$$

i.e. $\{K_i\}$ is a compact exhaustion of M.

Proof. Fix a countable basis $\mathcal{B}$ for M. For each $p \in M$, let U_p be a neighborhood of p with compact closure, and let B_p be a basis element with

$$p \in B_p \subset U_p.$$

Choose an enumeration of

$$\mathcal{B}' = \{B_p : p \in M\}$$

as

$$\mathcal{B}' = \{B_1, B_2, B_3, \dots\}$$

Then set

$$K = \bigcup_{j=1}^{i} \overline{B}_j$$

The $\overline{B}_j$ is a closed subset of some $\overline{U}_p$, which is compact, so itself is compact, and

$$\bigcup_i K_i = M.$$

■

Proof of Theorem. Let (K_i) be a compact exhaustion of M as in the Lemma.

PIC

Set

$$V_j = K_{j+1} \setminus \text{Int}(K_j)$$

so is compact; and

$$w_j = \text{Int}\, K_{j+2} \setminus K_{j-1}$$

so is open. If $p \in V_j$, pick some $A_p \in \mathcal{A}$ with $p \in A_p$ and then pick a basis element

$$B_p \in \mathcal{B}, \text{ and } p \in B_p \subset A_p \cap W_j.$$

Then the union of all such B_p covers V_j, which is compact, so there exists a finite subcover. Let $\mathcal{B}'$ be the union of all these finite subcovers for $j = 1, 2, 3, \dots$. Then $\mathcal{B}' \subset \mathcal{B}$ and is a locally finite refinement of $\mathcal{A}$. Check each of these. For local finiteness: Given $p \in M$, $p \in V_i$ for some i, and then W_i is a neighborhood of p. But the sets above only intersect W_i for

$$i - 2 \leq j \leq i + 2$$

and there are finitely many elements of $\mathcal{B}'$ associated to each j, so only finitely many intersect W_i. ■

Definition 1.4.11: Partition of Unity

Suppose M is a topological space, and $X = \{X_\alpha\}$ an open cover. A *partition of* *unity subordinate to* X is a family

$$\rho_\alpha \colon M \to [0,1]$$

satisfying

1. $\operatorname{supp} \rho_\alpha \subset X_\alpha$ for all α.
2. The set of supports $\{\operatorname{supp} \rho_\alpha\}$ is locally finite, i.e. every point of M has a neighborhood that intersects only finitely many supports.
3. $\sum_\alpha \rho_\alpha(x) = 1$ for all $x \in M$.

Theorem 1.4.12: Existence of Partitions of Unity

Suppose M is a topological manifold, with or without boundary. Take $X = \{X_\alpha\}$ an open cover. Then there exists a partition of unity subordinate to X. If M is smooth one can take the functions in the partitions of unity to be smooth.

Proof. Let's assume M is a smooth manifold without boundary. By the theorem from last time, there exists a locally finite refinement $\{U_i\}$ of X such that for each i, there exists some $V_i \supset U_i$ and a chart

$$\phi_i \colon V_i \to B_3(0) \subset \mathbb{R}^n$$

such that

$$U_i = \phi_i^{-1}(B_2(0)).$$

such U_i's form a basis for the topology of M. Let

$$H \colon \mathbb{R}^n \to [0,1]$$

be > 0 exactly on $B_2(0)$ and $= 0$ otherwise. Then set

$$f_i \colon M \to \mathbb{R}$$

defined by

$$f(x) = \begin{cases} H \circ \phi_i & x \in V_i \\ 0 & \text{otherwise} \end{cases}$$

Then this f_i is smooth, with support $\operatorname{supp} f_i = \overline{U_i}$. Since $\{U_i\}$ is locally finite, thus so is the set of supports

$$\{\operatorname{supp} f_i\} = \{\overline{U_i}\}.$$

This is because if a neighborhood $W \ni p$ intersects $\overline{U_i}$, it intersects U_i too. Each f_i is supported inside $\overline{U_i}$, which is contained in some X_α. Something went wrong here, see Lee ?

While probably $\sum f_i \neq 1$, not that each p has a neighborhood on which only finitely many f_i are non-zero (supports are locally finite), so

$$f(x) = \sum_i f_i(x)$$

is well defined and smooth. Also, $f > 0$ because $f_i > 0$ on U_i and the U_i cover M. So, if we set

$$g_i = \frac{f_i}{f}$$

then we have

1. $\operatorname{supp} g_i = \overline{U_i} \subset \text{some } X_\alpha$
2. $\{\operatorname{supp} g_i\}$ is locally finite
3. $\sum g_i = 1$

Problem: g_i's are indexed by i, not α. So, for each i, pick some $\alpha(i)$ such that

$$\overline{U_i} \subset X_{\alpha(i)}.$$

Then for a given α, set

$$\rho_\alpha = \sum_{i:\ \alpha(i)} g_i$$

We have $\operatorname{supp} \rho_\alpha = \bigcup_{i|\alpha(i)=\alpha} \overline{U_i} \subset X$ ■

Corollary 1.4.13: Existence of Smooth Bump Functions

Suppose M is a smooth manifold. For any closed $A \subset M$ and open $A \subset U$. Then there exists a smooth bump function for A supported in U. That is, there exists a smooth function $f\colon M \to [0,1]$ such that

$$f \equiv 1 \text{ on } A, \operatorname{supp} f \subset U.$$

Proof. Set $V = M \setminus A$, and let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$, where $\operatorname{supp} \rho_U \subset U, \operatorname{supp} \rho_V \subset V$. Then ρ_U has the desired properties: $\rho \equiv 1$ on A since $\operatorname{supp} \rho_V \subset V$ implies $\rho_V \equiv 0$ on A and we must have $\rho_U + \rho_V = 1$. ■

§1.4.1 Some Applications of Partitions of Unity

Definition 1.4.14: Proper Map

A map $f\colon M \to N$ is *proper* if preimages of compact sets are compact. i.e. whenever $K \subset N$ is compact, so is $f^{-1}(K)$.

Here's an example of a proper map

$$\mathbb{R}^n \to [0,\infty]$$
$$x \mapsto \|x\|$$

Indeed, if $K \subset [0,\infty]$ is compact, then $K \subset [0,r]$ for some $r < \infty$, thus $f^{-1}(K) \subset f^{-1}([0,r]) = \overline{B_r(0)} \subset \mathbb{R}^n$, which is compact. Since K is closed, and the function is continuous, the preiamge $f^{-1}(K)$ is a closed subset of the compact set $\overline{B_r(0)}$ and hence is compact.

how to interpret properness? Recall that the *1-point compactification* of a metric space X is the space

$$\hat{X} = X \cup \{\infty\}$$

where the topology is generated by open subsets of X, and sets of the form $(X \setminus K) \cup \{\infty\}$ where $K \subset X$ is compact.

A sequence (x_n) is X converges to ∞ in $\hat{X}$ is defined to be: for all $K \subset X$ compact, there exists $N \in \mathbb{N}$ such that $x_n \notin K$ for all $n \geq N$. We usually write $x_n \to \infty$; or say "x_n exists every compact subset of X", or "(x_n) exists X".

What's the point? $f\colon X \to Y$ is proper if and only if whenever $x_n \to \infty$ then $f(x_n) \to \infty$ as well. E.g. if $x_n \to \infty$ in $\mathbb{R}^n$, then the norm $|x_n| \to \infty$.

Here's the application of partitions of unity:

Corollary 1.4.15

If M is a smooth manifold (even with boundary), there exists a smooth proper function $f\colon M \to [0, \infty)$.

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Note: If M is compact, we can take f to be constant.

Remark 1.4.16

You can replace second-countability in the definition of a manifold with metrizability. This is naively related to the Corollary, in that you'd like to set $f(x) = d(x, p)$ for some fixed p. But this function may not be smooth, and may not be proper, e.g. if $M = (0, 1)$ with Euclidean distance.

Proof. Let $\{V_j\}$ be a countable open cover of M where each V_j has compact closure (invoking second-countability). Let $\{\rho_j\}$ a subordinate partition of unity. Set

$$\begin{aligned} f\colon M &\to [0, \infty) \\ p &\mapsto \sum_j j \cdot \rho_j(p) \end{aligned}$$

This f is smooth and positive. If $K \subset \mathbb{R}$ is compact, pick $r > 0$ such that $K \subset [-r, r]$. Then if $p \in f^{-1}(K)$, we have

$$f(p) = \sum_j j \cdot \rho_j(p) < r.$$

so some j with $\rho_j(p) \neq 0$ satisfies $j < r$. Thus $p \in \overline{V_j}$ for some j. So

$$f^{-1}(K) \subset \bigcup_{j=1}^{r} \overline{V_j}$$

which is compact. We also know $f^{-1}(K)$ is compact since f is smooth so is continuous. ■

§1.5 Lie Groups

Definition 1.5.1: Lie group

A *Lie group* is a smooth manifold G with a group structure such that the multiplication and inversion maps

$$\begin{aligned} m \colon G \times G &\to G \\ (a, b) &\mapsto ab \end{aligned}$$

$$\begin{aligned} i \colon G &\to G \\ a &\mapsto a^{-1} \end{aligned}$$

are smooth.

Example 1.5.2

1. $(\mathbb{R}^n, +)$

2. $$\mathrm{GL}_n(\mathbb{R}) = \{\text{invertible } n \times n \text{ matrices}\}$$

   is an open subset of $\{n \times n \text{ matrices}\} \cong \mathbb{R}^{n^2}$ and hence is a smooth manifold. Matrix multiplication is polynomial in the entries, so is smooth. Cramer's rule gives a formula for A^{-1} in terms of determinants of submatrices of A, and determinants are polynomials of the entries, so the entries of A^{-1} are rational functions (with denominator $\det A \neq 0$) in the entries of A, and hence smooth.

   Also, if V is a finite dimensional vector space, $\mathrm{GL}(V)$ is a Lie group.

3. $S^1 = \{e^{i\theta} \colon \theta \in \mathbb{R}\} \subset \mathbb{C}$ with complex multiplication. (You check)

4. If G and H are Lie groups, then $G \times H$ is.

5. Any discrete, countable group is a 0-dim Lie group.

Remark 1.5.3

If G is a Lie group, $g \in G$, let

$$\begin{aligned} L_g \colon G &\to G \\ L_g(x) &\mapsto gx \end{aligned}$$

be the "left translation by g" map. This map is smooth because multiplication is smooth, it also have an inverse $L_{g^{-1}}$, so is a diffeomorphism. If we let g act on G via L_g, we have a transitive action of G on itself by diffeomorphisms. So in particular, no manifold with non-empty boundary has a smoothly compatible group structure since there does not exist a diffeomorphism taking a boundary point into the interior.

Definition 1.5.4: Lie group homomorphism

If G, H are Lie groups, a Lie homomorphism is a smooth group homomorphism

$f\colon G \to H$.

Example 1.5.5

1. $S^1 \hookrightarrow \mathbb{C}^* = \mathbb{C} \setminus 0$.
2. $\exp\colon (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$ is a Lie group isomorphism.
3. $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_{\neq 0}$.
4. $f\colon \mathbb{R} \to S^1$, $f(t) = e^{2\pi i t}$.

Any topological manifold G with a continuous group structure admits a smooth structure with respect to which the operations are smooth, and even such a real analytic structure. Real analytic maps of $\mathbb{R}^n$ are those that are locally expressible as power series. A real analytic structure on a manifold is an atlas with real analytic transition maps. See Gleeson, Montgomery, Zippin 1952, answering Hilbert's 5th Problem.

Also, any continuous group homomorphism between Lie groups is smooth.

Chapter 2

Calculus on Manifolds

§2.1 From Multivariable Calculus

We start by doing some review of multivariable calculus. If $p \in \mathbb{R}^n$, the *tangent space at p is*

$$T\mathbb{R}^n_p := \mathbb{R}^n$$

viewed as the space of vectors based at p.

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We define

$$\begin{aligned} Head \colon T\mathbb{R}^n_p &\to \mathbb{R}^n \\ v &\mapsto v + p \end{aligned}$$

$$\begin{aligned} Vec_p \colon \mathbb{R}^n &\to T\mathbb{R}^n_p \\ x &\mapsto x - p \end{aligned}$$

Suppose $U \subset \mathbb{R}^n$ is open. Then $f \colon U \to \mathbb{R}^n$ is differentiable at $p \in U$ if there exists a linear map

$$\mathrm{d}f_p \colon T\mathbb{R}^n_p \to T\mathbb{R}^m_{f(p)}$$

called the *derivative map*, such that

$$\lim_{x \to p} \frac{|f(x) - Head(\mathrm{d}f_p(Vec_p(x)))|}{|x - p|}$$

In other words, the linear map $\mathrm{d}f_p$ approximates f up to first-order at p:

Example 2.1.1

If $L \colon \mathbb{R}^n \to \mathbb{R}^m$, and $p \in \mathbb{R}^n$, then L is differentiable at p with $\mathrm{d}L_p = L$.

Exercise: Show $\mathrm{d}f_p$ is unique if it exists.

If f is differentiable at p, then in coordinates, $\mathrm{d}f_p$ is represented by the Jacobian matrix

$$Jf_p = \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x^1} & \cdots & \frac{\partial f_m}{\partial x^n} \end{pmatrix}$$

where $f = (f_1, \ldots, f_m)$.

Example 2.1.2

$\gamma \colon (a, b) \to \mathbb{R}^n$ is differentiable at p if and only if the derivatives of all the components γ_i exist, in which case

$$d\gamma_t(1) = J\gamma_p = \begin{pmatrix} \frac{d\gamma_1}{dt} \\ \vdots \\ \frac{d\gamma_n}{dt} \end{pmatrix}$$

where $1 \in \mathbb{R}^n = T\mathbb{R}^n_t$. This is equal to

$$\lim_{s \to 0} \frac{\gamma(t+s) - \gamma(t)}{s} = \gamma'(t)$$

i.e. the velocity vector of γ at time t. Picture:
PIC

Remark 2.1.3

In higher dimensions, it is not true that f is differentiable when all partial derivatives exist, e.g.

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R} \\ (x.y) &\mapsto \begin{cases} 0 & x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

Then this has all partials defined at 0, but is not differentiable at 0.

Theorem 2.1.4

If f is C^1 in a neighborhood of p, i.e. all first partial derivatives exist and are continuous in a neighborhood of p, then f is differentiable at p.

Suppose $f\colon \mathbb{R}^n \supset U \to \mathbb{R}^m$ is differentiable at p, $v \in T\mathbb{R}^n_p$. Then $df_p(v)$ is called the *directional derivative of f in the direction of v*. What is it? Let us try approaching p along the path $t \mapsto p + tv$.

$$\lim_{t \to 0} \frac{|f(p+tv) - (df_p(tv) + f(p))|}{|tv|} = 0$$

then multiplying by $|v|$ and reorganizing:

$$\lim_{t \to 0} \left| \frac{f(p+tv) - f(p)}{t} - \frac{t df_p(v)}{t} \right| = 0$$

Thus

$$df_p(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

PIC

Example 2.1.5

If $v = e^i$, the ith standard basis vector, then

$$\mathrm{d}f_p(e^i) = \lim_{t\to 0} \frac{f(p+te^i) - f(p)}{t} = \begin{pmatrix} \frac{\partial f_1}{\partial x^i} \\ \vdots \\ \frac{\partial f_m}{\partial x^i} \end{pmatrix}$$

A Corollary of this is that $\mathrm{d}f_p$ is represented in coordinates by the Jacobian Jf_p, the matrix of partials. Here's the proof: The coordinate representation of $\mathrm{d}f_p$ is

$$(\mathrm{d}f_p(e^1), \cdots, \mathrm{d}f_p(e^n)) = Jf_p.$$

Example 2.1.6

Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x,y) = x^2 - y^2$.
PIC

$$Jf_{(0,0)} = (0,0), \quad Jf_{(1,0)} = (2,0)$$

so this is saying at $(0,0)$, f is well-approximated by the zero function; and at $(1,0)$ it is well-approximated by $(x,y) \mapsto 2x$.

Theorem 2.1.7: Chain rule

Suppose

$$f\colon \mathbb{R}^n \supset U \to \mathbb{R}^m, \quad g\colon \mathbb{R}^m \supset V \to \mathbb{R}^k$$

and $f(p) \in V$. If f, g are differentiable at $p, f(p)$ respectively, then $g \circ f$ is differentiable at p and

$$\mathrm{d}(g\circ f)_p = \mathrm{d}g_{f(p)} \cdot \mathrm{d}f_p.$$

Corollary 2.1.8

If $f\colon U \to V$ is a diffeomorphism, then

$$\mathrm{d}f^{-1}_{f(p)} = (\mathrm{d}f_p)^{-1}.$$

In particular this implies $\mathrm{d}f_p$ is invertible.

Proof. Apply chain rule to $f \circ f^{-1} = \mathrm{Id}$. ■

Corollary 2.1.9

If $f\colon \mathbb{R}^n \supset U \to V \subset \mathbb{R}^m$ is a diffeomorphism, then $m = n$.

Proof. $\mathrm{d}f_p\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism, so $m = n$. ■

§2.2 On Manifolds

We would like to define tangent spaces and derivatives for manifolds and smooth maps
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§2.3 Tangent Space

We want for every manifold M, point $p \in M$, a vector space TM_p, the *tangent space at* p. We also want for every $f\colon M \to N$ that is smooth at p, we want a linear map

$$\mathrm{d}f_p\colon TM_p \to TN_{f(p)}$$

such that

1. $T\mathbb{R}^n_p \coloneqq \mathbb{R}^n$, and $TH^n_p \coloneqq \mathbb{R}^n$ defined as before, and the derivative $\mathrm{d}f_p$ of a map $f\colon \mathbb{R}^n \supset U \to V \subset \mathbb{R}^m$ is as before (i.e. generalizing what we did before).
2. If $\mathrm{Id}\colon M \to M$ is the identity map, then $\mathrm{d}\,\mathrm{Id}_p = \mathrm{Id}\colon TM_p \to TM_p$ for all p.
3. (Locality) If $U \subset M$ is open and $i\colon U \hookrightarrow M$ is the inclusion, then

   $$\mathrm{d}i_p\colon TU_p \to TM_p$$

   is an isomorphism for all p.
4. (Chain rule) If $f\colon M \to N$, and $g\colon N \to P$ are smooth at $p, f(p)$ respectively, then

   $$\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$$

Properties 2 and 4 together imply that if $f\colon M \to N$ is a diffeomorphism, then $\mathrm{d}f_p$ is an isomorphism for all p.

Now if M is an n-manifold, say even with boundary, then TM_p is n-dimensional for all $p \in M$. Indeed, pick a chart $\phi\colon U \to \hat{U} \subset H^n$ around p. Then

$$M \hookleftarrow U \xrightarrow{\phi} \hat{U} \hookrightarrow H^n$$

taking derivatives:

$$TM_p \xleftarrow{\mathrm{d}i_p} TU_p \xrightarrow{\mathrm{d}\phi_p} T\hat{U}_{\phi(p)} \xrightarrow{\mathrm{d}i_p} TH^n_{\phi(p)} \cong \mathbb{R}^n$$

where these maps are isomorphisms. In the future, we will often use locality to identify $TU_p = TM_p$. In this case, you can interpret the above as saying that a chart induces an isomorphism

$$\text{“}\mathrm{d}\phi_p\colon TM_p \to TH^n_{\phi(p)}\text{”}$$

§2.3.1 A Construction of the Tangent Space at p

Pick a chart at p

$$\phi\colon U \to \hat{U}$$

and define $TM_p = T\mathbb{R}^n_{\phi(p)}$. But we don't want to have to pick a specific chart. So rigorously, set

$$TM_p \coloneqq \{(\phi, v)\colon \phi\colon U \to \hat{U} \subset \mathbb{R}^n \text{ is a chart around } p, v \in T\mathbb{R}^n_{\phi(p)}\} /\!\sim$$

where $(\phi, v) \sim (\psi, w)$ if

$$\mathrm{d}\big(\psi \circ \phi^{-1}\big)_{\phi(p)}(v) = w.$$

This is an equivalence relation: for instance if $(\phi, v) \sim (\psi, w)$ then

$$\begin{aligned}\mathrm{d}\big(\phi \circ \psi^{-1}\big)_{\psi(p)}(w) &= \mathrm{d}\big(\psi \circ \phi^{-1}\big)_{\psi(p)}(w) \\ &= \mathrm{d}(\psi \circ \phi^{-1})^{-1}_{\phi(p)}(w) \\ &= v.\end{aligned}$$

You can verify reflexivity and transitivity.

At this point the tangent space is not a vector space yet. For all charts ϕ, the map

$$\begin{aligned}T\mathbb{R}^n_{\phi(p)} &\to TM_p \\ v &\mapsto [(\phi, v)]\end{aligned}$$

is a bijection. It is injective because $(\phi, v) \sim (\phi, w)$ implies $w = \mathrm{d}\big(\phi \circ \phi^{-1}\big)(v)$ thus $v = w$. It is surjective because if $[(\psi, w)] \in TM_p$, then

$$(\psi, w) \sim (\phi, \mathrm{d}\big(\phi \circ \psi^{-1}\big)(w)).$$

We then define a vector space structure on TM_p so the above bijective maps are linear isomorphisms. That is, given two elements of TM_p, we can represent them as pairs $[(\phi, v)]$, $[(\phi, w)]$ and define

$$[(\phi, v)] + [(\phi, w)] = [(\phi, v + w)]$$

and similarly

$$\lambda[(\phi, v)] = [(\phi, \lambda v)].$$

This is well defined since $\mathrm{d}\big(\psi \circ \phi^{-1}\big)$ is linear.

Definition 2.3.1

If $f\colon M \to N$ is smooth at p, we define

$$\mathrm{d}f_p\colon TM_p \to TN_{f(p)}$$

by

$$\mathrm{d}f_p([\phi, v]) = [(\psi, \mathrm{d}\big(\psi \circ f \circ \phi^{-1}\big)_{\phi(p)}(v))]$$

where ϕ is a chart around p, and ψ is a chart of N around $f(p)$.

Exercise: Show well defined, and linear. The linearity follows from the fact that $\mathrm{d}\big(\psi \circ f \circ \psi^{-1}\big)$ is linear.

Now we must verify the properties we wanted in the beginning.

2. Identity Given $\mathrm{Id}\colon M \to M$, pick a chart ϕ around $p \in M$ and use it for both charts in the domain and range. So

$$\mathrm{d}\,\mathrm{Id}_p([\phi, v]) = [(\phi, \mathrm{d}\big(\phi \circ \mathrm{Id}\,\phi^{-1}\big)_p(v))] = [\phi, v]$$

3. Locality If $U \subset M$ is open, pick a chart ϕ_M for M around p and restrict it to give a chart ϕ_U for U. Then if $i\colon U \hookrightarrow M$ is the inclusion,

$$\begin{aligned} \mathrm{d}i_p([\phi_U, v]) &= [\phi, \mathrm{d}\big(\phi_M \circ i \circ \phi_U^{-1}\big)_{\phi_U(p)}(v)] \\ &= [\phi, v] \end{aligned}$$

thus $\mathrm{d}i_p$ is an isomorphism.

Try to verify the chain rule.

Theorem 2.3.2

There exists only one definition of $TM_p, \mathrm{d}f_p$ up to canonical isomorphism.

Proof. Homework. ■

§2.4 Derivations

Let M be an n-manifold and let

$$C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}.$$

This is an algebra over $\mathbb{R}$, i.e. elements of $C^\infty(M)$ can be scaled by real numbers, and they can be added and multiplied pointwise.

Let $v \in TM_p$, define the map

$$\begin{aligned} D_v\colon C^\infty(M) &\to \mathbb{R} \\ f &\mapsto \mathrm{d}f_p(v) \in T\mathbb{R}_{f(p)} \cong \mathbb{R} \end{aligned}$$

where we called $\mathrm{d}f_p(v)$ the derivative of f in the direction v (directional derivative).

Proposition 2.4.1

If $f, g \in C^\infty(M)$, then

1. $D_v(\lambda f) = \lambda \cdot D_v(f)$, for all $\lambda \in \mathbb{R}$.
2. $D_v(f + g) = D_v(f) + D_v(g)$.
3. (product/Leibniz rule) $D_v(f, g) = D_v f \cdot g(p) + f(p) \cdot D_v g$.

Proof. First, note that the proposition is true for $M = \mathbb{R}^n$:

1. Trivial.
2. Trivial.
3. This is the multivariable product rule, which can be proved by considering Jacobian matrices, which amounts to using the one-variable product rule to write out all the partial derivatives.

For the general case, choose a chart $\phi\colon U \to \hat{U} \subset \mathbb{R}^n$ around p. Then for all $f \in C^\infty(M)$,

$$\begin{aligned} D_v(f) &= \mathrm{d}f_p(v) \\ &= \mathrm{d}\big(f \circ \phi^{-1}\big)_{\phi(p)}(\mathrm{d}\phi_p(v)) \quad \text{chain rule backwards} \\ &= D_{\mathrm{d}\phi_p(v)}(f \circ \phi^{-1}). \end{aligned}$$

Moreover,

$$(f+g)\circ\phi^{-1} = f\circ\phi^{-1} + g\circ\phi^{-1}$$
$$(f\cdot g)\circ\phi^{-1} = (f\circ\phi^{-1})\cdot(g\circ\phi^{-1})$$

so

$$\begin{aligned} D_v(f+g) &= D_{\mathrm{d}\phi_p(v)}((f+g)\circ\phi^{-1}) \\ &= D_{\mathrm{d}\phi_p(v)}(f\circ\phi^{-1} + g\circ\phi^{-1}) \\ &= D_{\mathrm{d}\phi_p(v)}(f\circ\phi^{-1}) + D_{\mathrm{d}\phi_p(v)}(g\circ\phi^{-1}). \end{aligned}$$

And similarly for products and also for scaler multiplication. ■

Definition 2.4.2: Derivations at p

A *derivation* of $C^\infty(M)$ at p is a linear map

$$\Delta\colon C^\infty(M)\to\mathbb{R}$$

such that

$$\Delta(fg) = \Delta(f)g(p) + f(p)\Delta(g).$$

The set of all derivations at p is written DM_p and is a vector space under addition.

The Proposition above says that we have a map

$$\begin{aligned} TM_p &\to DM_p \\ v &\mapsto D_v \end{aligned}$$

This map is linear, since $D_{\alpha v+\beta w} = \alpha D_v + \beta D_w$ since $\mathrm{d}f_p$ is linear for every f.

Theorem 2.4.3

This map above $v\mapsto D_v$ is an isomorphism of vector spaces.

This gives another proof of the uniqueness of TM_p up to isomorphism. Alternatively, you can define TM_p as DM_p, like in Lee.

Before the proof, we need some facts about derivations. Let $\Delta\in DM_p$. Then

1. If f is constant, then $\Delta(f)=0$.
2. If $f(p)=g(p)=0$, then $\Delta(fg)=0$.
3. If $f=g$ in a neighborhood of p, then $\Delta(f)=\Delta(g)$.

Proof.

1. It suffices by linearity to show that $\Delta(1)=0$. We have

$$\begin{aligned} \Delta(1) &= \Delta(1\cdot 1) \\ &= \Delta(1)\cdot 1 + 1\cdot\Delta(1) \\ &= 2\Delta(1), \end{aligned}$$

   implying $\Delta(1)=0$.

2. This follows immediately from the product rule.

3. Suppose $\rho\colon M \to \mathbb{R}$ is a smooth function that vanishes outside U, and $\rho \equiv 1$ in the neighborhood of p where $f = g$ (its existence is guaranteed by the existence of smooth bump functions). Then

$$\begin{aligned} 0 &= \Delta(\rho \cdot (f-g)) \quad \text{since } 0 = (f-g)(p) \\ &= \Delta(f-g)\rho(p) + \Delta(\rho)(f-g)(p) \\ &= \Delta(f-g) \\ &= \Delta(f) - \Delta(g). \end{aligned}$$

■

§2.5 Tangent Vectors in Coordinates

In $\mathbb{R}^n$, we will sometimes use the notation $\frac{\partial}{\partial x^i}\big|_p$ for the element $e_i \in T\mathbb{R}^n_p$. The notation reflects that we can view e_i as the associated directional derivative, i.e. the ith partial.

If $\phi = (x^1, \dots, x^n)$ is a chart for M, we will also abusively write

$$TM_p \ni \frac{\partial}{\partial x^i}\bigg|_p := \mathrm{d}\phi_p^{-1}\left(\frac{\partial}{\partial x^i}\bigg|_{\phi(p)}\right) \in T\mathbb{R}^n_{\phi(p)}.$$

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The vectors $\frac{\partial}{\partial x^i}\big|_p$, $i = 1, \dots, n$ form a basis for TM_p.

§2.6 Velocity Vectors

Suppose $\gamma\colon (a,b) \to M$ is a smooth path. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \gamma'(t) := \mathrm{d}\gamma_t\left(\frac{\partial}{\partial t}\right) \in TM_{\gamma(t)}$$

is called the *velocity vector* of γ at time t.

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Proposition 2.6.1

Any $v \in TM_p$ is a velocity vector of some smooth path γ with $\gamma(0) = p$.

Proof. Pick a chart ϕ around p and take the path

$$\begin{aligned} \gamma\colon (-\epsilon, \epsilon) &\to M \\ t &\mapsto \phi^{-1}\underbrace{((\phi(p) + t\mathrm{d}\phi_p(v))}_{\text{line in } \mathbb{R}^n} \end{aligned}$$

which is defined for small ϵ. Then

$$\gamma(0) = \phi^{-1}(\phi(p)) = p$$

and

$$\begin{aligned}\gamma'(0) &= \mathrm{d}\gamma_0\left(\frac{\partial}{\partial t}\right)\\ &= \mathrm{d}\phi^{-1}_{\phi(p)}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\phi(p)+t\mathrm{d}\phi_p(v))\Big|_{t=0}\right)\\ &= \mathrm{d}\phi^{-1}_{\phi(p)}(\mathrm{d}\phi_p(v))\\ &= v.\end{aligned}$$

■

So we can now view tangent vectors as velocity vectors of paths. Also via the chain rule, if $v \in TM_p$ and γ satisfies $\gamma'(0) = v$, then for $f\colon M \to N$ some smooth function, we can visualize $\mathrm{d}f_p(v)$ as

$$\begin{aligned}\mathrm{d}f_p(v) &= \mathrm{d}f_p(\gamma'(0))\\ &= \mathrm{d}f_p\left(\mathrm{d}\gamma_0\left(\frac{\partial}{\partial t}\right)\right)\\ &= \mathrm{d}(f\circ\gamma)_0\left(\frac{\partial}{\partial t}\right)\\ &= (f\circ\gamma)'(0).\end{aligned}$$

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§2.7 The Tangent Bundle

Suppose M is a smooth manifold and set

$$TM = \bigsqcup_{p\in M} TM_p.$$

Sometimes we will write $v \in TM_p$ as (p, v). There is a natural projection

$$\begin{aligned}\pi\colon TM &\to M\\ (p,v) &\mapsto p\end{aligned}$$

Example 2.7.1

For $U \subset \mathbb{R}^n$,

$$TU = \bigsqcup_{p\in U} T\mathbb{R}^n_p \cong U \times \mathbb{R}^n.$$

Definition 2.7.2: Global differential

If $f\colon M \to N$ is smooth, set the *global differential* to be

$$\begin{aligned}\mathrm{d}f\colon TM &\to TN\\ (p,v) &\mapsto (f(p), \mathrm{d}f_p(v))\end{aligned}$$

Note that
$$\mathrm{d}(g \circ f) = \mathrm{d}g \circ \mathrm{d}f.$$

Theorem 2.7.3: Smooth structure on the tangent bundle

TM has a natural $2n$-dimensional smooth structure (where $n = \dim M$) such that the projection $\pi\colon TM \to M$ is smooth. Moreover, if TM, TN are equipped with their smooth structures and $f\colon M \to N$ is a smooth map, then $\mathrm{d}f\colon TM \to TN$ is smooth.

Note, given $f\colon M \to N$ smooth, we can then consider the "second derivative"
$$\mathrm{d}\mathrm{d}f\colon T(TM) \to T(TN)$$
but this is very confusing.

Proof. Suppose $\phi\colon M \supset U \to \hat{U} \subset \mathbb{R}^n$ be a chart for M. Then we can use
$$\mathrm{d}\phi\colon TM \supset TU \to T\hat{U} \cong \hat{U} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$$
as a chart for TM. If (ψ, V) is another chart for M, we have
$$\mathrm{d}\psi \circ \mathrm{d}\phi^{-1}\colon T\phi(U \cap V) \to T\psi(U \cap V)$$
but by the chain rule, this is just
$$\mathrm{d}\big(\psi \circ \phi^{-1}\big),$$
so in coordinates it is
$$\begin{aligned} \mathrm{d}\big(\psi \circ \phi^{-1}\big)(p, v) &= \big(\psi \circ \phi^{-1}(p), \mathrm{d}\big(\psi \circ \phi^{-1}\big)_p(v)\big) \\ &= \big(\psi \circ \phi^{-1}(p), J(\psi \circ \phi^{-1})_p \cdot v\big) \end{aligned}$$
The $J(\psi \circ \phi^{-1})_p$ is a matrix of partials varying smoothly with p, since $\psi \circ \phi^{-1}$ is smooth. So $\mathrm{d}\psi \circ \mathrm{d}\phi^{-1}$ is smooth. The charts $\mathrm{d}\phi$ for TM have open images in $\mathbb{R}^{2n}$, and any two points of TM are either in the same chart or in disjoint charts, and countably many of them cover TM. So by Lemma from before, they are charts in a unique smooth structure.

With respect to these charts, we have
$$\begin{array}{ccc} TM \supset TU & \xrightarrow{\mathrm{d}\phi} & T\hat{U} \subset \mathbb{R}^n \times \mathbb{R}^n \\ \big\downarrow \pi & & \big\downarrow p \\ M \supset U & \xrightarrow{\quad \phi \quad} & \hat{U} \subset \mathbb{R}^n \end{array}$$
where p is the projection onto the first $\mathbb{R}^n$, which is smooth. So π is smooth. And similarly, given $f\colon M \to N$, if $(U, \phi), (V, \psi)$ are charts around $p, f(p)$, respectively,
$$\begin{array}{ccc} TM \supset TU & \xrightarrow{\mathrm{d}f} & TN \subset TV \\ \big\downarrow \mathrm{d}\phi & & \big\downarrow \mathrm{d}\psi \\ T\hat{U} & \xrightarrow[\mathrm{d}(\psi \circ f\phi^{-1})]{} & T\hat{V} \end{array}$$
and since $\psi \circ f\phi^{-1}$ is smooth, so is $\mathrm{d}\big(\psi \circ f\phi^{-1}\big)$, this implies $\mathrm{d}f$ is smooth. ■

§2.8 Vector Fields

Definition 2.8.1

A *vector field* on M is a map $X\colon M \to TM$ such that $\pi \circ X = \mathrm{Id}$, i.e. $X(p) \in TM_p$ for all $p \in M$.

The smooth structure on TM allows one to say X is continuous or smooth, etc.

On $\mathbb{R}^n$, vector fields have the form

$$p \mapsto (p, X(p)), \quad X(p) \in \mathbb{R}^n$$

and we often write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}.$$

There will be more on vector fields on the homework.

Chapter 3

Structures of Smooth Manifolds

§3.1 Classes of Maps between Manifolds

Definition 3.1.1: Local diffeomorphism

A smooth map $f\colon M \to N$ is a *local diffeomorphism at* p if there exists open neighborhoods $U \ni p$ and $V \ni f(p)$ such that

$$f\big|_U : U \to V$$

is a diffeomorphism.

Example 3.1.2

1. Diffeomorphisms are local diffeomorphisms. This is trivial, just take $U = M$ and $V = N$.
2. The inclusion $i\colon U \to M$ of an open subset, take $U = U$, $V = i(U)$.
3. Smooth covering maps.

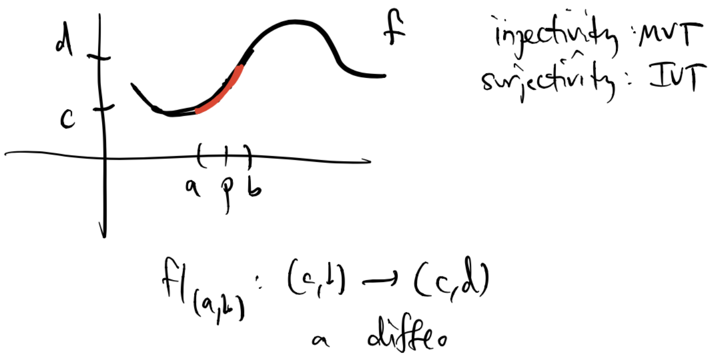
Note, if f is a local diffeomorphism at p, then $\mathrm{d}f_p$ is an isomorphism. Since derivatives of diffeomorphisms are isomorphisms by chain rule and only depend on f in a neighborhoood of p. In fact, we have the converse:

Theorem 3.1.3: Inverse Function Theorem

If $f\colon M \to N$ is a smooth map such that $\mathrm{d}f_p$ is an isomorphism, then f is a local diffeomorphism at p.

Remark 3.1.4

1. Suffices to prove for $\mathbb{R}^n \to \mathbb{R}^n$, by choosing charts.
2. For $f\colon \mathbb{R} \to \mathbb{R}$, the Inverse Function Theorem says that if $f'(p) \neq 0$ then f is a local diffeomorphism at p. This is because a non-zero linear map between one-dimensional vector spaces must necessarily be an isomorphism.



Definition 3.1.5: Contraction

Suppose X is a metric space. A map $g\colon X \to X$ is a *contraction* if there exists $\lambda < 1$ such that

$$d(g(x), g(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$.

Lemma 3.1.6

Suppose X is a complete metric space. Then every contraction $g\colon X \to X$ has a unique fixed point.

Proof of Lemma. Take $x \in X$. Then (we can show) the sequence $\{g^i(x)\}$ is Cauchy. Hence $g^i(x) \to p \in X$ as $i \to \infty$. It follows that p has to be a fixed point by continuity of g. Thus a fixed point exists.

If p, q are fixed points, and $p \neq q$, then

$$d(g(p), g(q)) = d(p, q) \not\leq \lambda d(p, q).$$

A contradiction. ■

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Proof of Inverse Function Theorem. Suffices to prove for $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $\mathrm{d}f_0 = \mathrm{Id}$.

Set $h(x) = f(x) - x$, so $\mathrm{d}h_0 = 0$. Pick $\epsilon > 0$ such that $|\mathrm{d}h_p| < \frac{1}{2}$ for all $p \in B_\epsilon(0) = B$. where the $|\mathrm{d}h_p|$ is the Euclidean norm of Jacobian, i.e.

$$\sqrt{\sum_{i,j} a_{ij}^2}.$$

If $x, y \in B$, by Prop. C29 in Lee, we have

$$|h(x) - h(y)| < \frac{1}{2}|x - y|$$

(an application of Mean Value Theorem essentially). But by triangle inequality and definition of h,

$$\begin{aligned}|x-y| &\le |f(x)-f(y)| + |h(x)-h(y)| \\ &\le |f(x)-f(y)| + \frac{1}{2}|x-y|\end{aligned}$$

thus

$$\frac{1}{2}|x-y| \le |f(x)-f(y)|.$$

This shows f is injective on B.

We claim that $f(B) \supset B_{\epsilon/2}(0)$: If $|y| < \frac{\epsilon}{2}$, we want $x \in B$ such that $f(x) = y$. Set

$$G(x) = -h(x) + y = x - f(x) + y.$$

So, $f(x) = y$ if and only if $G(x) = x$. If $|x| \le \epsilon$, we have

$$\begin{aligned}|G(x)| &\le |h(x)| + |y| \\ &\le \frac{1}{2}|x| + \frac{\epsilon}{2} \\ &\le \epsilon.\end{aligned}$$

Hence G sends the closed ball $\overline{B_\epsilon(0)}$ into itself. It is also a contraction since

$$|G(x) - G(x')| = |h(x) - h(x')| \le \frac{1}{2}|x - x'|.$$

Applying Contraction Mapping Lemma, there exists $x \in B$ such that $f(x) = y$.

So if we take

$$U = B_\epsilon(0) \cap f^{-1}(B_{\epsilon/2}(0))$$

then

$$f\big|_U : U \to B_{\epsilon/2}(0)$$

is a bijection. It is a homeo by

$$\frac{1}{2}|x-y| \le |f(x)-f(y)|$$

which implies

$$\left|f^{-1}(x) - f^{-1}(y)\right| \le 2|x-y|.$$

You can check it is a diffeomorphism by showing directly that $\mathrm{d}f^{-1}$ is the derivative of f^{-1}, from the definition. See Lee. ∎

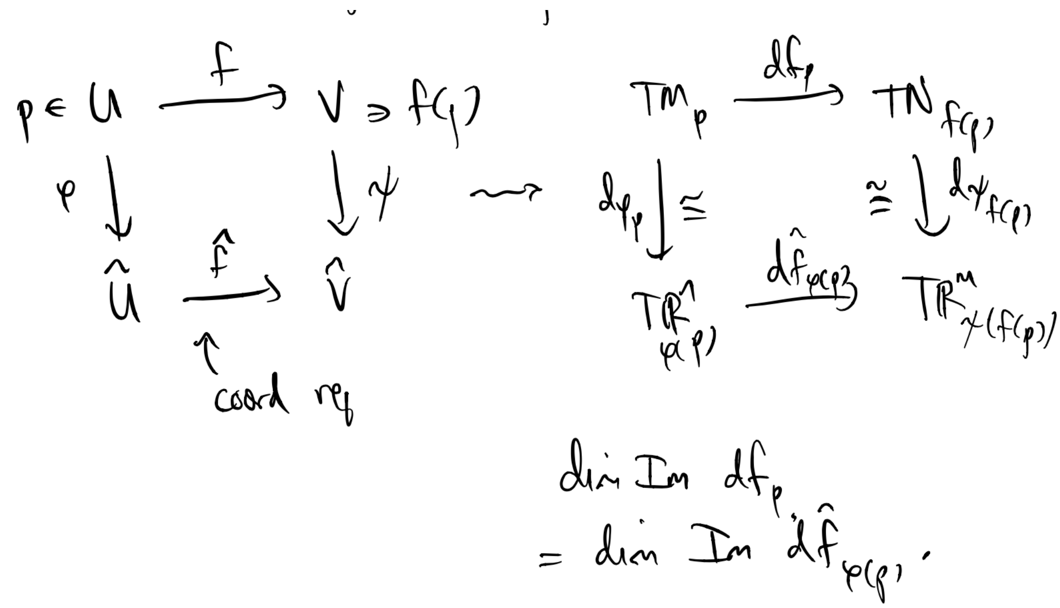
Definition 3.1.7

If $f\colon M \to N$ is smooth, the *rank of f at p* is defined to be

$$\dim \operatorname{im}(\mathrm{d}f_p).$$

If f has the same rank r at every point, we say it has *constant rank*, and we write $\operatorname{rank} f = r$.

[Rank of a Smooth Map] Note that, the rank of f at a point p is the rank of the coordinate representation of f at the image of p in the chart.



Proposition 3.1.8: $p \mapsto$ rank of f at p is lower semi-continuous

Suppose f has rank $\geq r$ at p. Then there exists a neighborhood of p on which f has rank $\geq r$.

In other words, we claim that the map $p \mapsto \operatorname{rank} f$ at p is lower semi continuous.

Proof. Suffices to take $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Then the condition $\operatorname{rank} \mathrm{d}f_p \geq r$ is equivalent to the statement that some $r \times r$ minor of Jf_p is non-zero. Here, a *minor* is the determinant of a square submatrix made by removing some of the rows and columns of Jf_p. This is an open condition on p, so $\operatorname{rank} \mathrm{d}f_q \geq r$ in a neighborhood of p. Since this specific minor of Jf_p is a continuous function of p, it is non-zero in a neighborhood of p, implying f has rank $\geq r$ in a neighborhood of p. ■

Definition 3.1.9: Map of Full Rank

We notice that the rank of f at p is always at most $\min\{\dim M, \dim N\}$. So if f has *full rank* (at p) if the rank equals this minimum (at p).

Corollary 3.1.10: Full Rankness is Local

If f has full rank at p, it has full (and in particular constant) rank in a neighborhood of p.

Definition 3.1.11: Submersion and Immersion

We say the smooth map $f\colon M \to N$ is a *submersion* (at p) if rank $= \dim N$ at p. We say f is an *immersion* (at p) if rank $= \dim M$ (at p).

Equivalently, f is a submersion at p if $\mathrm{d}f_p$ is surjective; and f is an immersion at p if $\mathrm{d}f_p$ is injective.

Example 3.1.12

A linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is

1. an immersion if and only if it is injective.
2. a submersion if and only if it is surjective.
3. a diffeomorphism if and only if it is bijective.

These are all consequences of the fact that $\mathrm{d}L_p = L$ for all p.

Example 3.1.13

1. If $\pi_M\colon M \times N \to M$, $\pi_M(p, q) = p$ is a submersion. In the usual charts for $M \times N$, π_M is projection onto the first coordinate, so its derivative is also, and hence has full rank.
2. $\pi\colon TM \to M$, same reason.
3. $\gamma\colon (a, b) \to M$ is an immersion if and only if $\gamma'(t) \neq 0$ for all t.

Theorem 3.1.14: Constant Rank Theorem

Suppose $f\colon M \to N$ has constant rank r in a neighborhood of p. Then there exists charts around $p, f(p)$ in which f has a coordinate representation of the following form:

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0).$$

When f is a submersion at p (so also in a neighborhood of p), this becomes

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^n).$$

When f is an immersion at p, this becomes

$$(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0).$$

Exercise: If $L\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear, show there exists isomorphisms $A\colon \mathbb{R}^m \to \mathbb{R}^m$ and $B\colon \mathbb{R}^n \to \mathbb{R}^n$ such that BLA^{-1} has the form above:

$$BLA^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0)$$

where $r = \dim L(\mathbb{R}^m) = \operatorname{rank} L$.

Proof. See Lee. ∎

Definition 3.1.15

An immersion $f\colon M \to N$ is called an *embedding* if it is a homeomorphism onto its image.

Example 3.1.16

1. The inclusion $i\colon U \hookrightarrow M$ of an open subset of M is an embedding.
2. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a homeomorphism onto its image since the topology on S^n is defined to be the subspace topology.

   To show immersion, using the natural coordinates

   $$\phi_i^+ = \text{projection onto the coordinate plane spaned by } e^1, \dots, \hat{e}_i, \dots, e_{n+1}$$

   we get

   $$i \circ (\phi_i^{\pm})^{-1}\colon B_1(0) \to \mathbb{R}^{n+1}$$

   is just $(\phi_i^{\pm})^{-1}$ which has full rank since it is

   $$(x^1, \dots, x^n) \mapsto (x^1, \dots, 1 - \sqrt{\sum x_i^2}, x^i, \dots, x^n)$$

   essentially because we see all the x^i's in the image.
3. If $f\colon M \to N$ is smooth, then $M \to M \times N$, $p \mapsto (p.f(p))$ is an embedding

Example 3.1.17: Non-examples

1. $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t^3, 0)$ is a homeomorphism onto its image but is not an immersion.
2. $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ as the following is not injective, so not an embedding.
3. $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ as the following is an injective immersion, but not a homeomorphism onto its image

   Let $T^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$, $\pi\colon \mathbb{R}^2 \to T^2$ the covering map, $\alpha \in \mathbb{R}$. Then

   $$\begin{aligned} \gamma_\alpha\colon \mathbb{R} &\to T^2 \\ t &\mapsto \pi(t, \alpha t) \end{aligned}$$

   PIC

   This γ_α is the composition of an immersion $t \mapsto (t, \alpha t)$ and a local diffeomorphism (covering), so it is an immersion.

Is it an embedding? If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then for all $n \in \mathbb{Z}$,

$$\begin{aligned}\gamma_\alpha(t+qn) &= \pi\left(t+qn, \frac{p}{q}(t+qn)\right) \\ &\pi(t+qn, \frac{p}{q}t+pn) \\ &= \pi(t, \frac{p}{q}t) \\ &= \gamma_\alpha(t)\end{aligned}$$

So γ_α is periodic with period q, thus γ_α is not an embedding since it is not injective. But it induces an embedding

$$S^1 \cong {}_{q\mathbb{Z}}\backslash^{\mathbb{R}} \to T^2.$$

If $\alpha \notin \mathbb{Q}$, then α is injective since suppose

$$\gamma_\alpha(t) = \gamma_\alpha(s)$$

where $t \neq s$. Then

$$(t-s, \alpha(t-s)) \in \mathbb{Z}^2$$

hence

$$\alpha(t-s) \in \mathbb{Z}$$

so $\alpha \in \mathbb{Q}$, a contradiction. But you can check that if $\alpha \notin \mathbb{Q}$, its image is a dense subset of T^2, so not an embedding (see Lee).

Example 3.1.18

The image of an embedding is called an *embedded submanifold.*

Remark 3.1.19: Smooth structure on embedded submanifold

Note: $N \subset M$ is an embedded submanifold if and only if it has a smooth structure such that the inclusion $i\colon N \hookrightarrow M$ is an immersion. The reverse direction is clear since i is always a homeomorphism onto its image. In the forward direction, if $f\colon N \to M$ is an embedding, we want to say $f(N)$ has a smooth structure such that the inclusion is an immersion. Idea is to use that f is a homeomorphism onto its image to transfer the smooth strcuture of N onto its image: if $\phi\colon U \to \hat{U}$ is a chart for N, let

$$\phi \circ f^{-1}\colon f(U) \to \hat{U}$$

be a chart for $f(N)$. This defines a smooth structure, and with respect to these charts, the coordinate representation of $i\colon f(N) \to M$ is just the coordinate representation of f, so i is an immersion since f was.

Theorem 3.1.20: Local Slice Criterion for Embedded Submanifolds

A subset $N \subset M$ is a k-dimensional embedded submanifold if and only if for each

$p \in N$ there exists an M-chart around p

$$\phi\colon U \to \hat{U} \subset \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$$

considered as

$$\phi = (\phi_1, \phi_2)$$

where ϕ_1 maps into $\mathbb{R}^k$ and ϕ_2 maps into $\mathbb{R}^{m-k}$. Such that

$$N \cap U = \phi^{-1}(\mathbb{R}^k \times \phi_2(p)).$$

This is called the *local slice condition.*

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Proof. Forward direction: Around p, the Local Immersion Theorem implies that there exists charts (U, ϕ) and (V, ψ) for N, M, respectively, sending $p \mapsto 0$ and where

$$\psi \circ i \circ \phi^{-1}(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0) \in \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

Shrink the domains of ϕ, ψ so their images are exactly the ϵ-balls around 0, for some small $\epsilon > 0$. Then

$$i(U) = \psi^{-1}(\mathbb{R}^k \times 0).$$

Moreover, since i is an embedding, there exists an open $W \subset M$ such that $W \cap N = i(U)$, since $i(U)$ is open in N. The restriction

$$\psi\big|_{V \cap W}\colon V \cap W \to \mathbb{R}^k \times \mathbb{R}^{m-k}$$

then satisfies

$$\left(\psi\big|_{V \cap W}\right)^{-1}(\mathbb{R}^k \times 0) = N \cap (V \cap W)$$

as desired. ■

Note: this Local Slice Condition is *not true* for the image of an immersion: PIC figure eight

or, even an injective immersion, e.g. an irrational line on the torus T^2.

Example 3.1.21

Suppose $f\colon \mathbb{R}^n \supset U \to \mathbb{R}^m$ is smooth. Then the graph

$$\Gamma(f) = \{(p, f(p)\colon p \in U\} \subset \mathbb{R}^n \times \mathbb{R}^m$$

satisfies the local slice condition, so is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$. This is because given $p \in U$, set

$$\phi\colon U \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$$

$$\phi(p, q) = (p, q - f(p)).$$

This is a chart for $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$\Gamma(f) = p^{-1}(\mathbb{R}^n \times 0).$$

Note, any subset of $\mathbb{R}^n$ that is locally the graph (of a smooth) of some coordinates against the other is similarly an (embedded) submanifold, e.g. $S^n \subset \mathbb{R}^{n+1}$.

Definition 3.1.22: Properly embedded submanifold

A *properly embedded submanifold* is an embedded submanifold $N \subset M$ such that $i\colon N \to M$ is proper. Equivalently, that N is the image of a proper embedding. Here, a map is proper if the preimages of compact sets are compact.

Example 3.1.23

1. Any compact submanifold, e.g. $S^n \subset \mathbb{R}^{n+1}$. Because any continuous map from a compact space to a Hausdorff space is proper.
2. (non-example) Take $(0,1) \subset \mathbb{R}$ is not properly embedded since $i^{-1}([0,1])$ is not compact.

Proposition 3.1.24

A submanifold $S \subset M$ is properly embedded if and only if S is closed in M.

Example 3.1.25

If M is a manifold with boundary, then the boundary $\partial M \subset M$ is a properly embedded submanifold. This is because M-charts all give the local slice condition, and ∂M is a closed subset.

Theorem 3.1.26

Suppose $f\colon M \to N$ is smooth with constant rank r. Then each *level set* $f^{-1}(q)$, $q \in N$, is a properly embedded submanifold of M with *codimension* r.

Here, if $X \subset M$ is a submanifold, then

$$\operatorname{codim} X \coloneqq \dim M - \dim X.$$

Proof. Around any point $p \in f^{-1}(q)$, the Constant Rank Theorem gives charts $(U,\phi),(V,\psi)$ around $p, f(p)$, respectively. Say with $\phi(p) = \psi(f(p)) = 0$, such that

$$\psi \circ f \circ \phi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

But then ϕ is a local slice chart for $f^{-1}(q) \subset M$ because

$$U \cap f^{-1}(q) = \phi^{-1}(0 \times \mathbb{R}^{m-r})$$

where $0 \times \mathbb{R}^{m-r}$ is exactly what maps to $0 = \psi(q)$ under $\psi \circ f \circ \phi^{-1}$. And the dimension of $f^{-1}(q) = r$.

The properness follows from f continuous, thus the preimages of points are closed. ■

Example 3.1.27: Constant rank maps

1. Any Lie homomorphism $f\colon G \to H$ (smooth homomorphism of Lie groups) has constant rank: If $g \in G$,

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ H & \xrightarrow[L_{f(g)}]{} & H \end{array}$$

   Then taking derivatives at the identity,

$$\begin{array}{ccc} TG_e & \xrightarrow{\mathrm{d}L_g} & G \\ {\scriptstyle \mathrm{d}f_e}\downarrow & & \downarrow{\scriptstyle \mathrm{d}f_g} \\ TH_e & \xrightarrow[\mathrm{d}L_{f(g)}]{} & TH_{f(g)} \end{array}$$

   Hence $\mathrm{d}f_e, \mathrm{d}f_g$ are maps with the same rank because the horizontal maps are isomorphisms. This implies f has constasnt rank. So the kernel $\ker f = f^{-1}(e)$ of any Lie homomorphism is a properly embedded submanifold of the domain.

2. $SL_n\mathbb{R} \subset M_{n\times n}$ is a properly embedded submanifold. $\mathrm{GL}_n\,\mathbb{R} \subset M_{n\times n}$ is an open submanidold.

$$\det \mathrm{GL}_n\,\mathbb{R} \to \mathbb{R}_{\neq 0}$$

   is a group homomorphism, soi has?? $\mathrm{SL}_n\,\mathbb{R}$ is a submanifold of $\mathrm{GL}_n\,\mathbb{R}$, hence of $M_{n\times n}$.

Definition 3.1.28: Regular/critical point/value

Suppose $f\colon M \to N$ is smooth. Then $p \in M$ is a *regular* point if $\mathrm{d}f_p$ is surjective, i.e. f is a submersion of p. We call p a *critical point* otherwise. We call $q \in N$ a *regular value* if each $p \in f^{-1}(q)$ is a regular point; otherwise, we call q a *critical value*.

Note, if $q \notin f(M)$, then q is a regular value.

Theorem 3.1.29

If $f\colon M \to N$ is smooth and $q \in N$ is a regular value, then $f^{-1}(q)$ is a properly embedded submanifold of M with codimension equal to the dimension of N.

Both the above Theorems imply that if f is a submersion, then each $f^{-1}(q)$ is a submanifold.

Proof. Same as before, using Local Submersion Theorem (Constant Rank Theorem applied to submersions). ■

Example 3.1.30

1. $S^n \subset \mathbb{R}^{n+1}$ is a submanifold of codimension 1. We can realize
$$S^n = f^{-1}(1)$$
where $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = |x|^2$. And $\mathrm{d}f_x = 0$ exactly when $x = 0$, and otherwise is surjective, so 1 is a regular value.

2. Set
$$O(n) = \{A \in M_{n\times n}\colon A^T A = I\}$$
be the *orthogonal group*. If $\langle,\rangle$ is the standard inner product (dot product) on $\mathbb{R}^n$, then
$$\begin{aligned} A \in O(n) &\Leftrightarrow \langle Av, Aw\rangle\langle v, w\rangle \\ &\Leftrightarrow \text{columns of } A \text{ form an ONB for } \mathbb{R}^n \\ &\Leftrightarrow \text{If } \{v_i\} \text{ is an ONB, so is } \{Av_i\}. \end{aligned}$$
Claim: $O(n)$ is a submanifold of $M_{n\times n}$. We will realize $O(n)$ as the preimage of a regular value of some map. Set
$$S(n,\mathbb{R}) = \{A \text{ a symmetric } n\times n \text{ matrix, i.e. } A^T = A\}$$
which is an $\frac{n(n+1)}{2}$-dimensional manifold, since we can prescribe arbitrarily all entries a_{ij} with $i \geq j$, and then the others are determined, so $S(n,\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$.

Define
$$\begin{aligned} \Phi\colon\ \mathrm{GL}_n\,\mathbb{R} &\to S(n,\mathbb{R}) \\ A &\mapsto A^T A \end{aligned}$$
We want to show
$$O(n) = \Phi^{-1}(I)$$
a submanifold, so we want to show I is a regular value: If $A \in O(n)$, let $\gamma(t) = A + tB$ where $B \in M_{n\times n}$. Then
$$\begin{aligned} \mathrm{d}\Phi_A(B) &= (\Phi\circ\gamma)'(0) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(A+tB)^T(A+tB)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}A^TA + tB^TA + tA^TB + t^2B^TB\Big|_{t=0} \\ &= B^TA + A^TB. \end{aligned}$$
Then
$$\mathrm{d}\Phi_A\colon T\,\mathrm{GL}_n\,\mathbb{R}_A \cong M_{n\times n} \to S(n,\mathbb{R}) \cong TS(n,\mathbb{R})_I$$

is surjective, since if $C \in S(m, \mathbb{R})$,

$$\begin{aligned} \mathrm{d}\Phi_A\left(\frac{1}{2}AC\right) &= \left(\frac{1}{2}AC\right)^T A + A^T\frac{1}{2}AC \\ &= \frac{1}{2}C^T(A^TA) + \frac{1}{2}C(A^TA) \\ &= \frac{1}{2}C + \frac{1}{2}C \\ &= C. \end{aligned}$$

§3.2 Sard's Theorem

Problem 2. $\mathbb{R}^n$ *comes equipped with the Lebesgue measure, which we denote by* vol. *Does a manifold come with a "Lebesgue measure"?*

However, there is a problem: transition maps may not preserve Lebesgue measure, e.g. if $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = 2x$ is a diffeomorphism but $\mathrm{vol}(f(A)) = 2^n \mathrm{vol}(A)$ for all measurable A, so vol is not preserved.

Lemma 3.2.1

uppose $f\colon \mathbb{R}^n \supset U \to \mathbb{R}^n$ is a smooth map and $A \subset U$ has measure zero, then $\mathrm{vol}(f(A)) = 0$ as well.

Recall: $A \subset \mathbb{R}^n$ has measure zero if and only if for all $\delta > 0$, there exists an open cover $\{U_i\}$ of A by open balls such that $\sum_i \mathrm{vol}(U_i) < \delta$.

Proof. It suffices to prove the Lemma when A is contained in a compact subset of $C \subset U$ (exhaust U by a sequence of compact subsets and use that countable unions of measure zero sets are measure zero). In that case, there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in C$ (Consequence of entries of Jacobian being bounded on compact subset C). Given $\delta > 0$, pick a cover of A by balls U_i with

$$\sum_i \mathrm{vol}(U_i) < \frac{\delta}{L^n}.$$

Then the sets $f(U_i)$ cover $f(A)$, and each is contained in a ball V_i of radius L (radius of U_i). So

$$\mathrm{vol}(V_i) \leq L^n \cdot \mathrm{vol}(U_i)$$

hence

$$\mathrm{vol}(f(A)) \leq L^n \cdot \sum_i \mathrm{vol}(U_i) < \delta.$$

Since $\delta > 0$ is arbitrary, $\mathrm{vol}(f(A)) = 0$. ■

Definition 3.2.2

If M is a smooth n-manifold, a subset $A \subset M$ has *measure zero* if $\phi(A \cap U)$ has measure zero in $\mathbb{R}^n$ for all charts ϕ.

Equivalently, if every point $p \in A$ is in the domain of a chart ϕ such that $\phi(A \cap U)$ has measure zero.

Here the equivalence if from the previous Lemma and the fact that unions of countably many measure zero sets are measure zero.

Theorem 3.2.3: Sard's Theorem

Suppose M, N are smooth manifolds, and $F\colon M \to N$ is smooth. Then the set $C \subset N$ of critical values of f has measure zero.

Recall: a critical point $p \in M$ is a point where $\mathrm{d}f_p$ is not surjective. A *critical value* is the image of a critical point. It is not true that the set of critical points in M has measure zero, e.g. if F is constant, then it is all of M.

Corollary 3.2.4

If $F\colon M \to N$ is smooth and $\dim M < \dim N$, then everything in the image is a critical value, hence $F(M)$ has measure zero. In particular, F is not surjective.

Remark 3.2.5

This Corollary is false for continuous maps, e.g. there exists surjective continuous maps $S^1 \to S^2$.

We will prove Sard for $F\colon \mathbb{R} \to \mathbb{R}$.

Sard's Theorem for $F\colon \mathbb{R} \to \mathbb{R}$. Set

$$C = \{\text{critical values of } f\} \subset \mathbb{R},$$

and

$$C_R = \{f(x)\colon x \in [-R, R] \text{ a critical point of } f\}$$

so that

$$C = \bigcup_{R=1}^{\infty} C_R.$$

Since this is a countable union, it suffices to show C_R has measure zero. Since f' is uniformly continuous on $[-R, R]$, given $\epsilon > 0$, there exists $\delta > 0$ such that if

$$|a - b| < 2\delta$$

where a is a critical point in $[-R, R]$ (so $f'(a) = 0$), we have

$$|f'(b)| < \epsilon.$$

Now cover $[-R, R]$ with $\leq \frac{4R}{\delta}$ number of δ-balls. For each $y \in C_R$, pick a critical point $x \in [-R, R]$ with $f(x) = y$, and let B be one of the above δ-balls containing x. Then $|f'| < \epsilon$ on B, then $f(B)$ is contained in a ball of radius $\epsilon \cdot \delta$, by MVT. As y varies, these $f(B)$ cover C_R. So we have

$$\mathrm{vol}(C_R) \leq \left(\frac{4R}{\delta}\right) \cdot 2\epsilon\delta = 8R\epsilon.$$

where $\frac{4P}{\delta}$ is the number of possible B's; and $2\epsilon\delta$ is the volume of interval of radius $\epsilon\delta$ containing $f(B)$. As ϵ was arbitrary, $\mathrm{vol}(C_R) = 0$. ■

§3.3 Application of Sard's Theorem

Theorem 3.3.1: Whitney's Embedding Theorem

If M is a smooth n-manifold, there exists a proper embedding $M \hookrightarrow \mathbb{R}^{2n+1}$

Remark 3.3.2

1. We will only prove the theorem when M is compact. See Lee for the general case.
2. Whitney later proved that M can be embedded in $\mathbb{R}^{2n}$ He also proved M can be immersed in $\mathbb{R}^{2n-1}$.
3. (Cohen) Any compact, smooth n-manifold can be immersed in $\mathbb{R}^{2n-a(n)}$, where $a(n)$ is the number of 1's in the binary expression of n.

Example 3.3.3

The Klein bottle has an immersion to $\mathbb{R}^3$. You can perturb this to an embedding $K \hookrightarrow \mathbb{R}^4$ by.

Proof. First, we show there exists an embedding $M \hookrightarrow \mathbb{R}^N$ for some N. Pick finitely many charts (using compactness)

$$\phi_i \colon U_i' \to \mathbb{R}^n, i = 1, \ldots, k$$

such that there exists U_i with $\overline{U}_i \subset U_i'$ such that the U_i's cover M and there are smooth functions ρ_i such that $\rho_i \equiv 1$ on u_i and ρ_i supported in U_i'. Set

$$\begin{aligned} F \colon M &\to \mathbb{R}^{nk+k} \\ x &\mapsto (\rho_1(x)\phi_1(x), \ldots, \rho_k(x)\phi_k(x), \rho_1(x), \ldots, \rho_k(x)) \end{aligned}$$

Note that each of the $\rho_i(x)\phi_i(x)$ are in $\mathbb{R}^n$, and each of the single $\rho_i(x)$'s are in $\mathbb{R}$. Here, we set $\rho_i(x)\phi_i(x) = 0$ outside U_i' ■

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Definition 3.3.4: Normal Space

If $M \subset \mathbb{R}^n$ is an m-dim embedded submanifold, then the *normal space* at $p \in M$ is

$$NM_p = (TM_p)^\perp \subset T\mathbb{R}^n_p \cong \mathbb{R}^n$$

The *normal bundle* is

$$NM = \bigsqcup_p NM_p \subset T\mathbb{R}^n$$

and we let

$$\pi\colon NM \to M$$

be the natural projection

$$\pi(x, v) = x.$$

Theorem 3.3.5: Dimension of the Normal Bundle

NM is an n-dim submanifold of the tangent bundle $T\mathbb{R}^n$.

Proof. By the local slice condition for embedded submanifolds, choose slice coordinate $\phi = (\phi^1, \dots, \phi^n)$ in some $U \subset \mathbb{R}^n$ such that

$$M \cap U = \{x\colon \phi^{m+1}(x) = \cdots = \phi^n(x) = 0\}$$

Now set for each $x \in U$,

$$\psi_x = (\psi_x^1, ; \psi_x^n)$$

where

$$\psi_x^i(v) = \left\langle \frac{\partial}{\partial x^i}|_x, v \right\rangle$$

coordinate system on $T\mathbb{R}^n_x$ ■

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Thinking of NM as a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, we define $E\colon NM \to \mathbb{R}^n$ by

$$E(x, v) = x + v.$$

Definition 3.3.6: Tubular neighborhood

Given a positive continuous function $\delta\colon M \to \mathbb{R}$, if the restriction of E to

$$V_\delta = \{(x, v) \in NM\colon |v| < \delta(x)\} \subset NM$$

is a diffeomorphism onto its image $U \subset r^n$ we call U a *tubular neighborhood*

Theorem 3.3.7

Every embedded submanifold $M \subset \mathbb{R}^n$ has a tubular neighborhood.

Proof. Let
$$M_0 = \{(x,0)\colon x \in M\} \subset NM$$
be the 0-*section*. Then $M_0 \subset NM$ is an n-dimensional submanifold (the charts from the previous proof can be used as slice charts). Then E restricted to M_0, $E\big|_{M_0}$ is a diffeomorphism onto $M \subset \mathbb{R}^n$, so

$$\mathrm{d}E_{(x,0)}((TM_0)_{(x,0)}) = TM_x.$$

We can view $((TM_0)_{(x,0)}) \subset T(NM)_{(x,0)}$. Also, for fixed $x \in M$, $NM_x \subset NM$ is an embedded submanifold

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and
$$E\big|_{NM_x}(x,v) = x + v$$
where x is fixed, so
$$\mathrm{d}E(T(NM_x)_{(x,0)}) = NM_x \subset T\mathbb{R}^n_x.$$
We can view $T(NM_x)_{(x,0)} \subset TNM_{(x,0)}$. So the image of $\mathrm{d}E$ contains

$$TM_x + NM_x = T\mathbb{R}^n_x.$$

This implies $\mathrm{d}E$ is an isomorphism at $(x,0) \in NM$, since $NM, \mathbb{R}^n$ both have dimension n. So, IFT implies E is a diffeomorphism onto its image when restricted to a neighborhood

$$V_\delta(x) = \{(x',v') \in NM\colon |x'-x| < \delta, |v'| < \delta\} \subset NM$$

for small δ. Let $\delta(x)$ be the supremum of all such δ. Then

$$\delta\colon M \to \mathbb{R}$$

is positive, and it is continuous, by triangle inequality (check). Set

$$V = \{(x,v)\colon |v| < \frac{1}{2}\delta(x)\}.$$

We want to show $E\big|_V$ is a diffeomorphism onto its image. We know it is a local diffeomorphism, hence it suffices to show that $E\big|_V$ is injective. If $(x,v),(x',v') \in V$ and

$$x + v = x' + v'$$

then

$$\begin{aligned}|x - x'| &= |x - v'| \\ &\leq |v| + |v'| \\ &\leq \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x') \\ &\leq \delta(x) \quad \text{WOLOG assume } \delta(x) \text{ is bigger}\end{aligned}$$

This is a contradiction, since then

$$(x,v),(x'v') \in V_{\delta(x)}(x)$$

map to the same thing, while E is supposed to be diffeo there (we used $|x-x'| < \delta(x)$ as above; and $|v|,|v'| < \frac{1}{2}\delta(x) < \delta(x)$ by defginition of V). ■

Proposition 3.3.8

If $U \supset M \subset \mathbb{R}^n$ is a tubular neighborhood of M, then there exists a submersion $r\colon U \to M$ that is a deformation retraction.

Proof. St $U \xrightarrow{E^{-1},\cong} V \subset NM \xrightarrow{\pi} M$, and let

$$r = \pi \circ E^{-1}.$$

This is a submersion. It is a deformation retract since $r = \mathrm{Id}$ on $M \subset U$, and the maps

$$U \xrightarrow{E^{-1}} V \xrightarrow{(x,v)\mapsto(x,tv)} V \xrightarrow{E} U$$

and let r_t be this composition. This gives a homotopy through U from $r_1 = \mathrm{Id}$ to $r_0 = r$. ∎

Theorem 3.3.9

Suppose M, N are smooth manifolds, $f\colon M \to N$ is a continuous map. Then f is homotopic to a smooth map g.

Lemma 3.3.10

If $f\colon M \to \mathbb{R}^n$ and $\epsilon\colon M \to \mathbb{R}_{>0}$ are continuous, there exists a smooth $g\colon M \to \mathbb{R}^n$ with

$$|f(x) - g(x)| < \epsilon(x)$$

for all $x \in M$.

Proof of Theorem. By Whitney Embedding, we may assume $N \subset \mathbb{R}^n$. Let $U \supset N$ be a tubular neighborhood, and pick $\epsilon\colon N \to \mathbb{R}_{>0}$ such that $B(y, \epsilon(y)) \subset U$ for all $y \in N$. By the Lemma, there exists a smooth

$$h\colon M \to \mathbb{R}^n$$

where

$$|f(x) - h(x)| < \epsilon(f(x)).$$

(the $\epsilon(x)$ in the Lemma is $\epsilon(f(x))$ here)

Note, if $x \in M$, then $h(x) \in U$ by the definition of ϵ.

PIC

Set

$$g = r \circ h\colon M \to N$$

where $r\colon U \to N$ is the submersion from the previous Proposition. Then $g\colon M \to N$ is smooth and

$$g_t(x) = r(\underbrace{(1-t)f(x) + th(x)U}_{\in U}).$$

This is a homotopy from $g = g_1$ to $f = g_0$. ∎

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§3.4 Vector Bundles

Definition 3.4.1: Vector bundle

If M is a smooth manifold, a *vector bundle of rank k over M* is a continuous map $\pi\colon E \to M$ such that:

1. For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ has the structure of a k-dimensional real vector space.

2. For each $p \in M$, there exists a neighborhood $U \subset M$ of p, and a homeomorphism
$$\phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$$
such that $\pi_U \circ \phi = \pi$ where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection (such ϕ's are called *local trivializations*); and for each $q \in U$, the restriction
$$\phi_q\colon E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$$
is a linear isomorphism.

If E and π are smooth, and the ϕ can be taken to be diffeomorphisms, we say $\pi\colon E \to M$ is a *smooth vector bundle.*

We call E the *total space*, M the *base space*, and π the *projection.*

Example 3.4.2

1. Take $M \times \mathbb{R}^k \to M$ is a smooth rank k vector bundle, which is trivial. We can use the identity map as a "local trivialization", which is really global.

2. (Tangent bundle) Let $\pi\colon TM \to M$, then this is a rank n vector bundle, where $n = \dim M$. If
$$\phi\colon U \to \hat{U} \subset \mathbb{R}^n$$
is a chart, then
$$\pi^{-1}(U) \cong TU \xrightarrow{\mathrm{d}\phi,\cong} T\hat{U} \cong \hat{U} \cong \mathbb{R}^n \xrightarrow{\phi^{-1}\times \mathrm{Id}} U \times \mathbb{R}^n$$
$$(p, v) \mapsto (p, \mathrm{d}\phi_p(v))$$
is a local trivialization.

3. (Normal bundle) Exercise. Use the slice charts we constructed for $NM \subset T\mathbb{R}^n$ to give local trivializations for $NM \to M$, where $m \subset \mathbb{R}^n$.

4. (Mobius bundle) Let $\mathbb{Z}$ act on $\mathbb{R}^2$, given by $n(x, y) = (x + n, (-1)^n y)$. Then set
$$M := \mathbb{Z} \backslash \mathbb{R}^2$$
one can verify this is the open Mobius band. We can see the fundamnetal domain as

PIC

The projection $\pi_{\mathbb{R}}(x,y) = x$ factors as

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\pi_M} & \mathbb{Z}\backslash\mathbb{R}^2 =: M \\ \downarrow{\scriptstyle \pi_{\mathbb{R}}} & & \downarrow{\scriptstyle \rho} \\ \mathbb{R} & \xrightarrow{\pi_{S^1}} & \mathbb{Z}\backslash\mathbb{R} =: S^1 \end{array}$$

PIC

This ρ is a rank 1 vector bundle. For a local trivilization, set $U = \pi_{S^1}(0,1)$ and set

$$\phi\colon U \times \mathbb{R} \to \rho^{-1}(U)$$

by setting

$$\phi(p,y) = \pi_M((\pi_{S^1}\big|_{(0,1)})^{-1}(p), y)$$

PIC This ϕ is a diffeo, and we can construct another such that

$$\psi\colon V = \pi_{S^1}\left(\frac{1}{2}, \frac{3}{2}\right) \times \mathbb{R} \to \rho^{-1}(V)$$

using the analogous formula. We want a vector space structure on each M_q, $q \in S^1$ such that ϕ, ψ give isomorphisms $\mathbb{R} \to M_q$. The "transition map" is

$$(0,1) \setminus \frac{1}{2} \times \mathbb{R} \to \rho^{-1}(U \cap V) \leftarrow (1/2, 3/2) \setminus 1 \times \mathbb{R}$$

which is $\psi^{-1} \circ \phi$

$$(x,y) \mapsto \begin{cases} (x,y) & x > \frac{1}{2} \\ (x+1,-y) & x < \frac{1}{2} \end{cases}$$

(check). So we can now equip each fiber $M_q = \rho^{-1}(q)$ with the unique vector space structure with respect to both ϕ, ψ are local trivilizations. On each fiber in the overlap, the transition map above is either $y \mapsto y$ or $y \mapsto -y$, which are both linear isomorphisms of $\mathbb{R}$, so the same vector space structure works.

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Supppose $\pi\colon E \to M$ is a smooth vector bundle and we have local trivializations

$$\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

and

$$\phi_\beta\colon \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$$

Then

$$\phi_\beta \circ \phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

has the form

$$\phi_\beta \circ \phi_\alpha^{-1}(p,v) = (p, \tau(p)v)$$

where $\tau(p) \in \mathrm{GL}_k(\mathbb{R})$. In other words, for each fixed p,

$$\phi_\beta \circ \phi_\alpha^{-1}\big|_{p \times \mathbb{R}^k}$$

is a linear isomorphism, hence is given by multiplication by an invertible matrix. Here

$$\tau_{\alpha\beta}\colon U_\alpha \times U_\beta \to \mathrm{GL}_k\,\mathbb{R}$$

is smooth, which is immediate from the fact that local trivializations are smooth (diffeo). These $\tau_{\alpha\beta}$'s are called *transition functions*.

Lemma 3.4.3: Vector Bundle Chart Lemma

Suppose M is a smooth manifold and for each $p \in M$, we have a k-dim vector space E_p. Let $E = \sqcup_p E_p$, let $\pi\colon E \to M$ be the obvious map, and suppose we have a cover $\{U_\alpha\}$ of M and for each α we have a bijection

$$\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

that is a linear isomorphism on every fiber, i.e.

$$\phi_\alpha\Big|_{E_p} E_p \to p \times \mathbb{R}^k$$

is an isomorphism, and where for each pair α, β, we have

$$\begin{aligned} \phi_\beta \circ \phi_\alpha^{-1}\colon U_\alpha \cap U_\beta \times \mathbb{R}^k &\to U_\alpha \cap U_\beta \times \mathbb{R}^k \\ (p, v) &\mapsto (p, \tau_{\alpha\beta}(p)v) \end{aligned}$$

for some smooth

$$\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_k \mathbb{R}.$$

Then there exists a unique smooth structure on E that makes $\pi\colon E \to M$ into a smooth vector bundle where the ϕ_α's are local trivializations.

Proof. See Lee. ■

Example 3.4.4: Whitney sums

Suppose $\pi_E\colon E \to M$, $\pi_{E'}\colon E' \to M$ are vector bundles of rank k, k', respectively. Set

$$\pi_F\colon F \to M$$

to be the union

$$F = \sqcup_p E_p \oplus E'_p$$

with π_p the obvious projection. If

$$\phi\colon \pi_E^{-1}(U) \to U \times \mathbb{R}^k, \phi = (\pi_E, \phi_2)$$

$$\phi'\colon \pi_{E'}^{-1}(U') \to U' \times \mathbb{R}^k, \phi' = (\pi_{E'}, \phi'_2)$$

then we set

$$\begin{aligned} \phi \oplus \phi'\colon \pi_F^{-1}(U \cap U') &\to \pi_F^{-1}(U \cap U') \\ (p, v + w) &\mapsto (p, \phi_2(p, v) + \phi'_2(p, w)) \end{aligned}$$

If $\tau_{\alpha\beta}$ is the transition function from ϕ_α to ϕ_β, and $\tau'_{\alpha\beta}$ is the transition function from ϕ'_α to ϕ'_β, then the transition function from $\phi_\alpha \oplus \phi'_\alpha$ to $\phi_\beta \oplus \phi'_\beta$ is

$$p \mapsto \begin{pmatrix} \tau_{\alpha\beta}(p) & 0 \\ 0 & \tau'_{\alpha\beta}(p) \end{pmatrix} \in \mathrm{GL}_{k+k'} \mathbb{R}$$

So there exists unique smooth structure on $E \oplus E' \coloneqq F$ making it into a vector bundle.

Example 3.4.5

Let $\pi\colon E \to M$ be a vector bundle, let $S \subset M$ be some submanifold. Then we can make a vector bundle

$$\pi_S\colon E\big|_S \to S$$
$$\left(E\big|_S\right)_p \coloneqq E_p$$

where we can take restrictions of local trivilizations for E as local trivilizations for $E\big|_S$.

Definition 3.4.6: Section of a vector bundle

A *section* of a vector bundle $\pi\colon E \to M$ is a map $\sigma\colon M \to E$ such that $\pi \circ \sigma = \mathrm{Id}$. So $\sigma(p) \in E_p$ for all p. We will assmu all sections are continuous, and offen we will consider smooth sections.

A *local section* over an open set $U \subset M$ is a section of $E\big|_U$, i.e. a map $\sigma\colon u \to E$ with $\pi \circ \sigma = \mathrm{Id}$.

Example 3.4.7

1. The zero section $\zeta\colon M \to E$, $\zeta(p) = (p, 0) \in E_p$.
2. Sections of $TM \to M$ are vector fields.
3. Sections of $M \times \mathbb{R}^k \to M$ are essentially functions $M \to \mathbb{R}^k$.

Definition 3.4.8: Local frame

Let $\pi\colon E \to M$ be a rank k vector bundle. A *local frame over* $U \subset M$ is a tuple

$$(\sigma_1, \dots, \sigma_k)$$

of local sections over U such that for all p,

$$\{(\sigma_1(p), \dots, \sigma_k(p))\}$$

is a basis for E_p. It is a *global frame* if $U = M$.

Local frames are "the same as" local trivializations: If $\phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization, we can let, for $p \in U$,

$$\sigma_i(p) = \phi^{-1}(p, e_i)$$

where e_i is the ith standard basis vector, and then the $\sigma_1, \dots, \sigma_k$ give a local fram over U.

Conversely, if $(\sigma_1, \dots, \sigma_k)$ is a local frame over u, the map

$$\pi^{-1}(U) \to U \times \mathbb{R}^k$$
$$\left(p, \sum_i a_i \sigma_i(p)\right) \mapsto (p, a_1, \dots, a_k)$$

is a local trivialization.

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Example 3.4.9

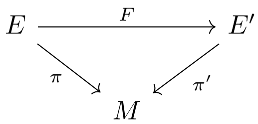
If $\phi \colon U \to \hat{U} \subset \mathbb{R}^n$ is a chart for M then the vector fields

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}$$

form a local frame for TM over U

Definition 3.4.10: Bundle homomorphism over a common base

Let $\pi \colon E \to M$ and $\pi' \colon E' \to M$ be smooth vector bundles over M. A *smooth bundle homomorphism over* M is a smooth map $F \colon E \to E'$ such that



commutes, i.e. $F(E_p) \subset E'_p$, and where

$$F_p \coloneqq F\big|_{E_p} \colon E_p \to E'_p$$

is linear.

We say F is an *isomorphism* if it is also a diffeomorphism, in which case each F_p is a linear isomorphism (because it is a bijection).

Definition 3.4.11: Trivial bundle

We say a bundle $\pi \colon E \to M$ is *trivial* if it is a isomorphic to a product bundle $M \times \mathbb{R}^k$.

Note, a local trivilization

$$\phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$$

is a bundle isomorphism from the restricted bundle $E\big|_U$ to $U \times \mathbb{R}^k$, hence the name local trivialization.

In particular, a bundle $E \to M$ is trivial if and only if there exists a global frame.

Example 3.4.12

The Mobius bundle $M \to S^1$ is not trivial because it admits no non-vanishing section, and hence has no global frame. Here a non-vanishing section is a $\sigma\colon S^1 \to M$, $\sigma(p) \neq 0 \in M_p$ for all p.

PIC

Any section of M gives a function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = -f(1)$. By the Intermediate Value Theorem, such a function must vanish somewhere.

Example 3.4.13

Suppose $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ are bundles, then the projection

$$E \oplus E' \to E$$

is a bundle homomorphism over M.

Definition 3.4.14: Bundle homomorphism, general

If $\pi\colon E \to M$, and $\pi'\colon E' \to M'$ are bundles over different spaces, and $f\colon M \to M'$ is smooth, a *smooth bundle homomorphism over* f is a smooth map $F\colon E \to E'$ such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

commutes; and where

$$F_p \coloneqq F\big|_{E_p}\colon E_p \to E'_{f(p)}$$

is linear for all p.

Example 3.4.15

If $f\colon M \to M'$ is smooth, then $\mathrm{d}f\colon TM \to TM'$ is a bundle homomorphism over f.

Case Study: Suppose $\pi\colon E \to M$ is a vector bundle, we can define $\pi^*\colon E^* \to M$ to be the *dual bundle*, where

$$(E^*)_p = (E_p)^* \coloneqq \{\text{linear } f\colon E_p \to \mathbb{R}\}$$

Here, if $e_1, \ldots, e_n$ is a local frame for E, then $e_1^*, \ldots, e_n^*$ is a local frame for E^*, where the e_i^*'s are defined by

$$e_i^*(e_j) = \delta_{ij}.$$

As an exercise, one can show that the associated local trivializations for E^* have transition functions of the form

$$\tau^*(p) = (\tau(p)^{-1})^T$$

hence are smooth maps into $GL_n\mathbb{R}$.

Definition 3.4.16: Cotangent bundle

The dual $T^*M = (TM)^*$ is called the *cotangent bundle*.

If $f\colon M \to \mathbb{R}$ is smooth, then the map

$$p \mapsto (\mathrm{d}f_p\colon TM_p \to T\mathbb{R}_{f(p)} = \mathbb{R}) \in T^*M_p$$

is a smooth section of the cotangent bundle T^*M, i.e. a *covector field* or a *1-form.* If we pick local coordinates $(x^1, \ldots, x^n)$ on $U \subset M$,

$$\mathrm{d}x^i_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta_{ij} \quad \text{for all } p$$

so $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ is the dual local frame to $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.

Hence, any 1-form, i.e. smooth section of T^*M can be written as

$$\omega = \sum_i a_i \mathrm{d}x^i$$

where smoothness of ω is equivalent to the smoothness of the coordinate functions a_i.

For instance,

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

The $\frac{\partial f}{\partial x^i}$'s are partials of the coordinate representation, or one can regard as $\mathrm{d}f\left(\frac{\partial}{\partial x^i}\right)$.

Chapter 4

Towards a Cohomology Theory for Smooth Manifolds

Goal for the rest of the course: We will construct a chain complex

$$0 \to \underbrace{C^\infty(M)}_{\text{"0-forms"}} \xrightarrow{d} \underbrace{\{1\text{-forms}\}}_{\text{sections of } T^*M} \xrightarrow{d} \underbrace{\{2\text{-forms}\}}_{\text{sections of some other bundle}} \xrightarrow{d} \dots$$

and a related *DeRham cohomology theory*,

$$H^k_{dR}(M) = \ker d\big|_{\{k\text{-forms}\}} \Big/ \operatorname{im} d\big|_{\{(k-1)\text{-forms}\}}$$

where $H^k_{dR}(M)$ will be a real vector space.

For instance,

$$\begin{aligned} H^0_{dR}(M) = \ker[d\colon C^\infty(M) \to \{1\text{-forms}\}] \\ &= \{\text{locally constant functions } f\colon M \to \mathbb{R}\} \\ &\cong \mathbb{R}^{\#\text{components}}, \end{aligned}$$

which is the same as $H_0(M;\mathbb{R})$ or $H^0(M;\mathbb{R})$, the singular (co)homology spaces.

§4.1 Linear Algebra and Tensors

Definition 4.1.1: Multilinear function

If V is a vector space, a function

$$T\colon \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

is *multilinear* if it is linear in each coordinate, i.e.

$$T(v_1, \dots, \alpha v_i + \beta w_i, \dots, v_k) = \alpha T(v_1, \dots, v_i, \dots, v_k) + \beta T(v_1, \dots, w_i, \dots, v_n)$$

for all α, β, v_j's, and w_i's.

We denote

$$T^k(V) := \{\text{multilinear functions } V^k \to \mathbb{R}\}.$$

Example 4.1.2

1. If $k = 1$, then multilinear means linear, so $T^1(V) = V^*$.

2. If $\phi_1, \dots, \phi_k \in V^*$, we can define the multilinear function

$$\phi_1 \otimes \cdots \otimes \phi_k \colon V \times \cdots \times V \to \mathbb{R}$$

defined by

$$\phi_1 \otimes \cdots \otimes \phi_k(v_1, \dots, v_k) \coloneqq \phi_1(v_1) \cdots \phi_k(v_k).$$

We call this map $\phi_1 \otimes \cdots \otimes \phi_k$ the *tensor product of* $\phi_1, \dots, \phi_k$. With this in mind, we often call elements of $T^k(V)$ *(contravariant) k-tensors.* Remark: In the more abstract language,

$$T^k(V) \cong V^* \otimes \cdots \otimes V^*,$$

and Lee calls it $T^k(V^*)$ instead of $T^k(V)$.

Note that $T^k(V)$ is a vector space because we can scale and add multilinear functions.

Theorem 4.1.3: A Basis for $T^k(V)$

Let V be an n-dim vector space. Suppose $\phi_1, \dots, \phi_n$ is a basis for the dual space V^*. Then

$$\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_k} : i_1, \dots, i_k \in \{1, \dots, n\}\}$$

is a basis for $T^k(V)$. Hence,

$$\dim T^k(V) = n^k.$$

Proof. Let $e_1, \dots, e_n$ be the basis for V dual to $\phi_1, \dots, \phi_n$; i.e. $\phi_i(e_j) = \delta_{ij}$.

For linear independence, it suffices to show no $\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$ is in the span of the others. To that end, we notice that

$$\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}(e_{i_1}, \dots, e_{i_k}) = 1 \cdots 1 = 1$$

but any other of the proposed basis elements gives 0 on $e_{i_1}, \dots, e_{i_k}$. Evaluation on $(e_{i_1}, \dots, e_{i_k})$ is a linear map $T^k(V) \to \mathbb{R}$, and it is non-zero exactly on the element $\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$ (among the proposed basis vectors), this shows that$\phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$ is not in the span of the other proposed elements.

Now for span of the whole space. Given an element $f \in T^k(V)$, i.e. a multilinear function $f \colon V^k \to \mathbb{R}$, we claim that

$$f = \sum_{\substack{\text{tuples} \\ (i_1, \dots, i_k)}} f(e_{i_1}, \dots, e_{i_k}) \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}.$$

Indeed, both sides give the same output when applied to $(e_{i_1}, \dots, e_{i_k})$; combining this with multilinearity implies both sides give the same output on all $(v_1, \dots, v_k)$. ■

Definition 4.1.4: Tensor Product of Two Multilinear Functions

If $T \in T^k(V)$, and $S \in T^\ell(V)$, then we can define the *tensor product of T and S*, which is a $(k+\ell)$-multilinear map

$$T \otimes S \in T^{k+\ell}(V)$$

defined by

$$T \otimes S(v_1, \dots, v_{k+\ell}) \coloneqq T(v_1, \dots, v_k) S(v_{k+1}, \dots, v_\ell).$$

This turns the sum of all the $T^k(V), k \geq 1$, i.e. $\bigcup_{k=1}^{\infty} T^k(V)$ into an object called the *tensor algebra*, in which you can sum multilinear maps using the vector space structure and multiply using the tensor product.

Definition 4.1.5: Alternating Multilinear Function

An element $T \in T^k(V)$ is *alternating* if

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

for all $v_1, \dots, v_k \in V$.

Theorem 4.1.6: Characterization of Alternating Multilinear Functions

Let $T \in T^k(V)$. The following are equivalent:

1. T is alternating.
2. $T(v_1, \dots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$.
3. If $a \in S_k$ (the symmetric group on k letters) acts on V^k by permuting the entries, then
$$T \circ \sigma = \operatorname{sgn}(\sigma) T$$
for any $\sigma \in S_k$.

Recall that the sign $\operatorname{sgn}(\sigma)$ of a permutation σ is defined to be $\operatorname{sgn}(\sigma) \coloneqq (-1)^s$ where s is the number of transpositions when we write $\sigma = \sigma_1 \cdots \sigma_s$, each σ_i being a transposition.

Definition 4.1.7: The Set of Alternating k-Multilinear Functions, Wedge-k of V

We denote the set of all alternating k-multilinear functions/alternating k-tensors to be

$$\Lambda^k(V)$$

We call this *Wedge-k of V*

Example 4.1.8

- 1-tensors are all alternating, so $\bigwedge^1(V) = T^1(V) = V^*$.
- $\mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto xy$ is a non-alternating 2-tensor, while
$$(x, y) \mapsto 0$$
is alternating.

Proposition 4.1.9

A k-tensor T is alternating if and only if $T(v_1, \ldots, v_k) = 0$ whenever $v_1, \ldots, v_k$ are linearly dependent.

Proof. First we assume that T is such that $T(v_1, \ldots, v_k) = 0$ whenever $v_1, \ldots, v_n$ is linearly dependent. We want to show T is alternating, which by the above Theorem, it suffices to whot that $T(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. But his follows immediately from the fact that if $v_i = v_j$ then the input set is linearly dependent, hence by the assumption, $T(v_1, \ldots, v_n) = 0$. So T is alternating.

Conversely, suppose T is alternating. Suppose we have $v_1, \ldots, v_k$ is a linearly dependent set. Then some v_i is a linear combination of the others, write it that way and expand using biliearity. All terms then have a repeated input, giving zero. ■

Corollary 4.1.10

$\Lambda^k(V) = 0$ if $k > n = \dim V$

Proof. If $k > n$, then there is no linearly independent set $v_1, \ldots, v_k$ of k vectors, so T always outputs zero. ■

Definition 4.1.11: Elementary Alternating k-Tensors

Fix a basis $\{e_1, \ldots, e_n\}$ for V and let $\{\epsilon^1, \ldots, \epsilon^n\}$ be the dual basis for V^*. For each k-multi-index

$$I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}$$

set

$$\epsilon^I(v_1, \ldots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \ldots & \epsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \epsilon^{i_k}(v_1) & \ldots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

These are called *elementary alternating k-tensors.*

Observe that ϵ^I is an alternating k-tensor, hence $\epsilon^I \in \Lambda^k(V)$.

When $I = \{1, \ldots, n\}$, ϵ_I is just the determinant map:

$$(v_1, \ldots, v_n) \mapsto \det(v_1, \ldots, v_n)$$

where we view the v_i's on the RHS as column coordinate vectors w.r.t. the e_i basis.

We say the k-multi-index $I = (i_1, \ldots, i_k)$ is *increasing* if $i_1 < i_2 < \cdots < i_k$.

Proposition 4.1.12

For a k-multi-index $J = (j_1, \ldots, j_k)$, where each $j_i \in \{1, \ldots, n\}$, set

$$e_J \coloneqq (e_{j_1}, \ldots, e_{j_k}).$$

Then if we have k-multi-indices $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_k)$, where both are increasing, then

$$\epsilon^I(e_J) = \begin{cases} 1 & I = J \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $I = J$, then

$$\epsilon^I(e_J) = \det(\text{Id}) = 1.$$

Otherwise, since both are increasing and length k, there is an index of J that is not an index of I, giving a zero column in our matrix, hence $\epsilon^I(e_J) = 0$. ■

Theorem 4.1.13: A Basis for Alternating k-Tensors

The set of alternating elementary k-tensors with increasing index:

$$\{\epsilon^I : I \text{ increasing, length } k\}$$

is a basis for $\Lambda^k(V)$.

Proof. Linear independence follows from the previous Proposition: suppose by way of contradiction that some ϵ^I in the set is the linear combination of some other ones, i.e. we can write

$$\epsilon^I = \sum_j a_j \epsilon^{I_j}$$

where all the I_j's $\neq I$. Then plugging e_I into both sides give $1 = 0$, a contradiction.

For span, suppose $\alpha \in \Lambda^k(V)$, then we observe that

$$\alpha = \sum_{\text{increasing } I} \alpha(e_I)\epsilon^I.$$

Indeed, both sides are alternating, and evaluate to the same thing on all e_J's, where J increasing, and hence on all e_J's (not necessarily increasing. This follows from both sides being alternating). Hence on all V^k by bilinearity. ■

Corollary 4.1.14

1. $\dim \Lambda^k(V) = \binom{n}{k}$.
2. $\Lambda^n(V) \cong \mathbb{R}$, spanned by the single element ϵ^I, where $I = \{1, \ldots, n\}$, i.e. the map
$$(v_1, \ldots, v_n) \mapsto \det(v_1, \ldots, v_n).$$

Remark 4.1.15: Determinant is the Unique Alternating Multilinear Map

It follows from this Corollary that the determinant is the unique function from the set of $n \times n$ matrices $M_{n\times n}$ to $\mathbb{R}$ that is alternating, multilinear in the columns, and satisfies $\det \text{Id} = 1$. Indeed, any function that is alternating and multilinear must be some constant multiple of the determinant function, and hence there is only one that satisfies $\det \text{Id} = 1$.

The pullback of an alternating k-tensor Suppose $L\colon V \to W$ is a linear map between vector spaces. Then there is an induced linear map

$$L^*\colon \Lambda^k(W) \to \Lambda^k(V)$$

called the *pullback*, defined by

$$L^*(T)(v_1, \dots, v_k) := T(L(v_1), \dots, L(v_k)).$$

In the special case where the linear map is $L\colon V \to V$, and say $n = \dim(V)$, we have

$$L^*\colon \Lambda^n(V) \cong \mathbb{R} \to \mathbb{R} \cong \Lambda^n(V)$$

is a linear map from $\mathbb{R}$ to $\mathbb{R}$, hence must be multiplication by a scalar. What is this scalar?

Choosing coordinates, and regarding the the following v_i's as column vectors, we have

$$\begin{aligned} L^* \det(v_1, \dots, v_n) &= \det(L(v_1), \dots, L(v_n)) \\ &= \det(L \cdot (v_1, \dots, v_n)) \quad \text{(here we regard } L \text{ as a matrix)} \\ &= \det(L) \cdot \det(v_1, \dots, v_n). \end{aligned}$$

Hence we see that the scalar is $\det(L)$.

Definition 4.1.16: $\mathrm{Alt}(T)$

Suppose $T \in T^k(V)$, we define

$$\mathrm{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T \circ \sigma.$$

Example 4.1.17

1. For $T \in T^1(V)$, we have $\mathrm{Alt}(T) = T$.
2. For $T \in T^2(V)$, we have

$$\mathrm{Alt}(T)(v, w) = \frac{1}{2}(T(v, w) - T(w, v)).$$

Proposition 4.1.18

1. $\mathrm{Alt}(T)$ is alternating.
2. If T is alternating, then $\mathrm{Alt}(T) = T$.

Proof. 1. Suppose τ is a transposition, then

$$\begin{aligned} \mathrm{Alt}(T) \circ \tau &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T \circ \sigma \circ \tau \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} -\mathrm{sgn}(\sigma \circ \tau) T \circ \sigma \circ \tau \\ &= -\mathrm{Alt}(T) \quad \text{reindexing} \end{aligned}$$

2. Homework.

■

This Proposition shows that we get a inear projection

$$\mathrm{Alt}\colon T^k(V) \to \Lambda^k(V).$$

Definition 4.1.19: Wedge Product

Suppose $\omega \in \Lambda^k(V), \eta \in \Lambda^\ell(V)$, define the *wedge product* of ω and η to be

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+\ell}(V).$$

Recall that the tensor product is defined to be

$$\omega \otimes \eta(v_1, \dots, v_{k+\ell}) = \omega(v_1, \dots, v_k) \cdot \eta(v_{k+1}, \dots, v_{k+\ell})$$

This defines a bilinear map

$$\wedge\colon \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V).$$

Proposition 4.1.20

If $\epsilon^1, \dots, \epsilon^n$ is a basis for V^*, dual to $e_1, \dots, e_n \in V$. Then

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$$

where IJ is the multi-index that is the concatenation of I and J.

Recall here that if we have the k-multi-index $I = (i_1, \dots, i_k)$, then we define

$$\epsilon^I(v_1, \dots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \dots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \dots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

Proof. Let $P = (p_1, \dots, p_{k+\ell})$ be a $(k+\ell)$-multi-index. Apply both sides to

$$e_P = (e_{p_1}, \dots, e_{p_{k+\ell}}).$$

If P contains a repeated index or an index not in either I or J, then both sides of the equation give 0 when applied to e_P. Hence, it suffices to take $P = IJ$, since we know how alternating k-tensors behave under permutation. We have

$$\begin{aligned}
\epsilon^I \wedge \epsilon^J(e_{IJ}) &= \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(\epsilon^I \otimes \epsilon^J)(e_{IJ}) \\
&= \frac{(k+\ell)!}{k!\ell!} \frac{1}{(k+\ell)!} \sum_{\sigma in S_{k+\ell}} \mathrm{sgn}(\sigma)(\epsilon^I \otimes \epsilon^J) \circ \sigma(e_{IJ}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)\epsilon^I(e_{\sigma(I)})\epsilon^J(e_{\sigma(J)})...
\end{aligned}$$

■

Proposition 4.1.21

The operation
$$\wedge\colon \Lambda^k(V) \times \Lambda^\ell(V) \to \Lambda^{k+\ell}(V)$$
is

1. Bilinear.
2. Associative: $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$.
3. Anticommutative: if $\omega \in \Lambda^k, \eta \in \Lambda^\ell$, then
$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$
4. If $I = (i_1, \dots, i_k)$ then
$$\epsilon^I = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}.$$
5. If $\omega^1, \dots, \omega^k \in V^*$, then
$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det\bigl(\omega^j(v_i)\bigr)$$
where $\omega^j(v_i)$ is the i, j-th entry of this $k \times k$ matrix.

Proof. ■

Remark 4.1.22

$$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$$
is called the *exterior algebra of* V. It is a graded associative algebra, in the sense that it is a vector space with a multiplication $\wedge$ that respects the "grading", i.e.
$$\Lambda^k \wedge \Lambda^\ell \subset \Lambda^{k+\ell}.$$
Hence, we define $\Lambda^0(V) \coloneqq \mathbb{R}$, and if $c \in \mathbb{R}$, then
$$c \wedge \omega \coloneqq c\omega.$$
Note also that if $L\colon V \to W$ is a linear map, we get an induced map
$$\Lambda(W) \to \Lambda(V)$$
and
$$L^*(\omega \wedge \eta) = L^* \left(\frac{(k+\ell)!}{k!\ell!} \operatorname{Alt}(\omega \otimes \eta) \right) = L^*\omega \wedge L^*\eta$$
that is, L^* is a map of algebras.

§4.2 Differential Forms

Let M be a smooth manifold. Set

$$\Lambda^k(TM) := \bigsqcup_p \Lambda^k(TM_p)$$

Now we shall give $\Lambda^k(TM)$ a smooth vector bundle structure (over M). Pick a chart with local coordinates $(x^1, \ldots, x^n)$ for M, and use

$$\{\mathrm{d}x^I : I \text{ an increasing } k\text{-multiindex}\}$$

as a local frame for $\Lambda^k(TM)$. Here, if we have the multiindex $I = (i_1, \ldots, i_k)$, then

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Recall here that dx^i is the dual to the basis element x^i, or $\frac{\partial}{\partial x^i}$.

Note that the dx^i form a basis for $(TM_p)^*$ at every p (these are the ϵ^i's), hence their wedges (i.e. the $e^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$ for I increasing) give a basis for $\Lambda^k(TM_p)$ at every p (see the Theorem on Basis for Alternating k-tensors).

Small example: $\Lambda^1(TM) = T^*M$.

Definition 4.2.1: Differential form

A smooth global section of $\Lambda^k(TM)$ is called a *(smooth) k-form*. We denote the set of all k-forms on M to be $\Omega^k(M)$.

Differential form in local coordinates In local coordinates, a k-form ω can be written as

$$\omega = \sum_{I \text{ increasing multiindex}} \omega_I \mathrm{d}x^I = \sum_I{}' \omega_I \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

where the ω_I's are smooth functions.

Definition 4.2.2: Wedge product of forms

We can define

$$\wedge \colon \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$$

pointwise. That is, we have

$$(\omega \wedge \eta)_p(v) = (\omega_p \wedge \eta_p)(v)$$

Definition 4.2.3: Pullback of forms

Suppose $F \colon M \to N$ is a smooth map between smooth manifolds. Let $\omega \in \Omega^k(N)$, we can define the *pullback of ω along F*: $F^*(\omega) \in \Omega^k(M)$, which is a k-form on M. It is defined by the following:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_k)).$$

Here are some facts about the pullback (Lemma 14.16 in Lee):

1. $F * \omega$ is smooth, and $\omega \mapsto F^*\omega$ is $\mathbb{R}$-linear.
2. $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. This is true because it is true pointwise(?).
3. In any smooth chart,

$$F^*\left(\sum_I{}' \omega_I \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, \mathrm{d}\big(y^{i_1} \circ F\big) \wedge \cdots \wedge \mathrm{d}\big(y^{i_k} \circ F\big)\,.$$

Example 4.2.4

Set $F(r, \theta) = (r\cos\theta, r\sin\theta)$, and let $\omega = \mathrm{d}x \wedge \mathrm{d}y$, which is a 2-form on $\mathbb{R}^2$.

$$\begin{aligned} F^*\mathrm{d}x \wedge \mathrm{d}y &= \mathrm{d}(r\cos\theta) \wedge \mathrm{d}(r\sin\theta) \\ &= (\cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta) \wedge (\sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta) \\ &= (\cos^2\theta \cdot r + \sin^2\theta \cdot r)\mathrm{d}r \wedge \mathrm{d}\theta \\ &= r\mathrm{d}r \wedge \mathrm{d}\theta. \end{aligned}$$

In the computation above,

$$JF = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

so

$$(\cos^2\theta \cdot r + \sin^2\theta \cdot r) = \det JF.$$

In general, if $F \colon U \to V$, where both $U, V \subset \mathbb{R}^n$, and $\omega = a\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ is an n-form on V, then

$$(f^*\omega_p = (\mathrm{d}f_p)^*\omega_{f(p)} = (\det Jf_p) \cdot a(f(p))\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)$$

by our earlier formula for pullbacks on $\bigwedge^n(V)$ by linear $L \colon V \to V$.

Proposition 4.2.5: Pullback of top degree forms

§4.3 Exterior Derivative

Exterior derivative on $\mathbb{R}^n$ If $\omega = \sum_I' \omega_I \mathrm{d}x^I$ is a k-form on $\mathbb{R}^n$, define the $(k+1)$-form

$$\mathrm{d}\omega := \sum_I{}' \mathrm{d}\omega_I \wedge \mathrm{d}x^I.$$

For example:

1. When ω is a 0-form, i.e. a smooth function, then $\mathrm{d}\omega$ is as before,

$$\mathrm{d}\omega = \sum_i \frac{\partial \omega_i}{\partial x^i}\mathrm{d}x^i.$$

2. If ω is a 1-form, then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\left(\sum_i \omega_i \mathrm{d}x^i\right) \\ &= \sum_i \mathrm{d}\omega_i \wedge \mathrm{d}x^i \\ &= \sum_i \left(\sum_j \frac{\partial \omega_i}{\partial x^j}\mathrm{d}x^j\right) \wedge \mathrm{d}x^i \\ &= \sum_{i<j}\left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right)\mathrm{d}x^i \wedge \mathrm{d}x^j. \end{aligned}$$

For instance, if ω is the derivative of some function,

$$\omega = \mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i,$$

then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f\mathrm{d}f \\ &= \sum \\ &= 0 \quad \text{by Clairault's Thm.} \end{aligned}$$

Proposition 4.3.1

1. d is linear over $\mathbb{R}$.
2. $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
3. $\mathrm{d} \circ \mathrm{d} \equiv 0$.
4. If $F\colon U \to V \subset \mathbb{R}^n$, ω a smooth k-form on V, then

$$F^*(\mathrm{d}\omega) = \mathrm{d}F^*\omega.$$

Proof. For 2., Suffices to take $\omega = u\mathrm{d}x^I, \eta = v\mathrm{d}x^J$. Then

$$\begin{aligned} \mathrm{d}(\omega \wedge \eta) &= \mathrm{d}\big(uv\mathrm{d}x^I \wedge \mathrm{d}x^J\big) \\ &= (v\mathrm{d}u + u\mathrm{d}v) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\ &= (\mathrm{d}u \wedge \mathrm{d}x^I) \wedge v\mathrm{d}x^J + (-1)^k u\mathrm{d}x^I \wedge (\mathrm{d}v \wedge \mathrm{d}x^J) \end{aligned}$$

Exercise: Show that for any multiindex I, $\mathrm{d}\big(a\mathrm{d}x^I\big) = \mathrm{d}a \wedge \mathrm{d}x^I$, not just for increasing I. ■

Exterior derivative on smooth manifolds

Theorem 4.3.2

Suppose M is a smooth manifold. Then there exist unique linear maps

$$\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$$

such that

1. $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
2. $\mathrm{d} \circ \mathrm{d} = 0$.
3. If $f \in \Omega^0(M) = C^\infty(M)$, then $\mathrm{d}f$ is the usual differential.

Moreover, in any coordinate chart, d is given by the formula we have for exterior differentialtion on $\mathbb{R}^n$, i.e.

$$\mathrm{d}\omega := \sum_I{}' \mathrm{d}\omega_I \wedge \mathrm{d}x^I.$$

Also, d commutes with pullbacks by smooth maps, i.e. if $f\colon M \to N$ is smooth, then

$$\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega.$$

§4.4 Orientation

Definition 4.4.1: Orientation on Vector Space

We say that two ordered bases for a vector space V *have the same orientation* if the change-of-basis matrix has positive determinant.

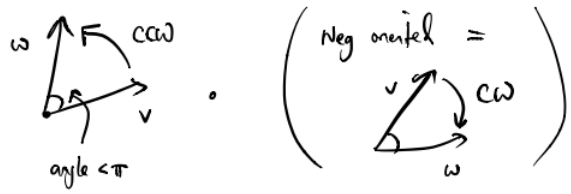
Equivalently, two ordered bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ have the same orientation if the unique linear map such that $L(e_i) = f_i$ for all $1 \le i \le n$ has positive determinant.

This then defines an equivalence relation on the set of ordered bases of V, with exactly two equivalence classes, called *orientations* of V. An *oriented vector space* is one equipped with a choice of orientation. If V is oriented, ordered bases in the chosen orientation are called *positively-oriented*, while the others are called *negatively-oriented*.

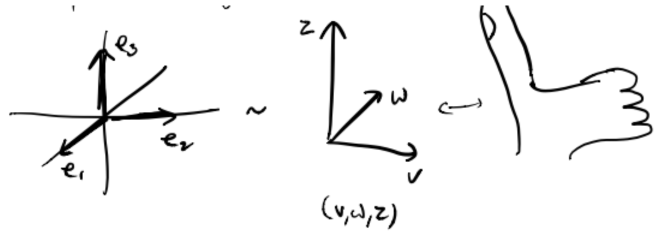
Example 4.4.2: Orientation on Euclidean Spaces

The *standard orientation of* $\mathbb{R}^n$ is the equivalence class of the standard basis $e_1, \ldots, e_n$.

In $\mathbb{R}^2$, equipped with the standard orientation, a basis v, w is positively-oriented if and only if the angle from v to w is counterclockwise; while the basis is negatively-oriented if the angle from v to w is clockwise.



In $\mathbb{R}^3$ (again with the standard orientation), positively-oriented bases are determined by the right-hand-rule:



Proposition 4.4.3: Orientation on Vector Space determined by a Tensor (Lee Prop.15.3)

Suppose V is an n-dimensional vector space. Then any non-zero $0 \neq \omega \in \Lambda^n(V)$ (this is an alternating n-tensor, so its a function on an n-tuple of vectors) determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(v_1, \ldots, v_n)$ such that $\omega(v_1, \ldots, v_n) > 0$; while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$.

Moreover, two elements of $\Lambda^n(V)$ determine the same orientation if and only if each is a positive multiple of the other.

Proof. If $(v_1, \ldots, v_n), (w_1, \ldots, w_n)$ are ordered bases and $A\colon V \to V, A(v_i) = w_i$. Then

$$\begin{aligned}\omega(w_1, \ldots, w_n) &= \omega(Av_1, \ldots, Av_n)\\ &= A^*\omega(v_1, \ldots, v_n) \quad \text{by definition of pullback of alternating tensor}\\ &= \det A \cdot \omega(v_1, \ldots, v_n). \quad \text{we showed that pulling back a top-deg tensor is multiplication by } \det A\end{aligned}$$

Hence, two bases have the same orientation if and only if $\omega(v_1, \ldots, v_n)$ and $\omega(w_1, \ldots, w_n)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily. Indeed,

Suppose $\omega, \eta \in \Lambda^n(V)$ determine the same orientation, i.e. $\omega(w_1, \ldots, w_n) > 0$ if and only if $\eta(w_1, \ldots, w_n) > 0$. Then ■

Remark 4.4.4

1. An orientation of a 1-dim vector space V is just a choice of a nonzero element up to a positive-multiple. The Proposition then implies that to give an orientation of V is equivalent to giving an orientation of $\Lambda^n(V) \cong \mathbb{R}$ (not sure what this means exactly).

2. An orientation of a 0-dim vector space V is either $+$ or $-$. Since $\Lambda^0(V) := \mathbb{R}$, this still corresponds to picking an orientation of $\Lambda^0(V)$.

3. A linear isomorphism $L\colon V \to W$ of oriented vector spaces if either *orientation preserving (o.p.)* or *orientation reversing (o.r.)*, depending on whether L sends positively-oriented bases to positively-oriented bases, or to negatively-oriented bases.

   To determine which one L is: pick positively-oriented bases for V and W, then look at the matrix representation A for L. If $\det A > 0$, then L is o.p.. Otherwise, L is o.r..

Definition 4.4.5: Pointwise Orientation on Manifold

A *pointwise orientation* for a manifold M is an orientation for each $TM_p, p \in M$.

For instance, $\mathbb{R}^n$ comes with a *standard orientation* on each $T\mathbb{R}^n_p = \mathbb{R}$.

We say that a local diffeomorphism $f\colon M \to N$ between pointwise oriented manifoldsd is *orientation preserving (o.p.)* or *orientation reversing (o.r.)* if $\mathrm{d}f_p$ is o.p. or o.r., for all $p \in M$. Note: a local diffeomorphism need not be either.

Definition 4.4.6: Orientation on Manifold

An *orientation* of a smooth manifold M is a pointwise orientation such that M is covered by orientation preserving charts into $\mathbb{R}^n$ (equipped with the standard orientation).

Remark 4.4.7

1. Would also suffice to say M is covered by orientation reversing charts, since you can compose with an orientation reversing diffeomorphism of $\mathbb{R}^n$, e.g. a reflection to get an orientation preserving atlas.
2. If $f\colon M \to N$ is a local diffeomorphism of oriented manifolds (not pointwise oriented) and M is connected, then f is either o.p. or o.r. (Exercise)
3. Equivalently, a pointwise orientation is an orientation if around every point, there is a local frame that is positively-oriented.

Proposition 4.4.8: The Orientation Determined by a Coordinate Atlas (Lee Prop. 15.6)

Suppose M has a smooth atlas where all transition maps are o.p. Then there exists a unique orientation of M such that the charts in the atlas are o.p..

Proof. To define the orientation on M, pick some $p \in M$ and a chart ϕ around p, and use $\mathrm{d}\phi_p$ to pullback the standard orientation on $\mathbb{R}^n$ to an orientation for TM_p. That is, let

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

be a positively-oriented basis for TM_p (recall: $\frac{\partial}{\partial x^i} = \mathrm{d}\phi_p^{-1}(e_i)$). Since transition maps are o.p., this is well-defined and the charts ϕ are o.p. ■

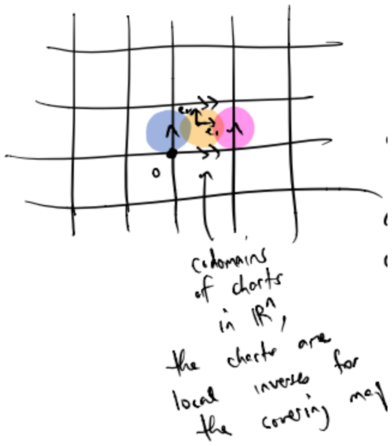
Definition 4.4.9: Oriented Manifold

A manifold is said to be *oriented* if it admits an orientation.

Example 4.4.10: T^n is an orientable manifold

We show that $T^n = \mathbb{Z}^n \backslash \mathbb{R}^n$ is orientable. The transition maps for the natural atlas coming from the covering map are restrictions of deck transformations, i.e.

translations, which are o.p.. The induced orientation of T^n is the one where $\mathbb{R}^n \to T^n$ is o.p..



Example 4.4.11: The Mobius Band is not orientable

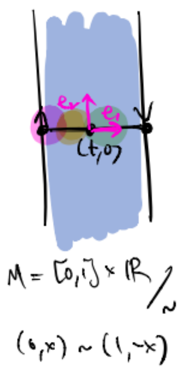
In the coordinates below, let

$$\epsilon \colon [0,1] \to \{+,-\}$$

where $\epsilon(t)$ is the sign of the basis $(e_1, e_2) \in TM_{(t,0)}$ with respect to an orientation we suppose we have. This $\epsilon(t)$ is continuous (i.e. constant) but $\epsilon(0) = -\epsilon(1)$ since $(0,0) \sim (0,1)$ and $T\mathbb{R}^2_{(0,0)}$ is identified with $T\mathbb{R}^2_{(0,1)}$ via the map

$$(x,y) \mapsto (x,-y),$$

which is orientation reversing.



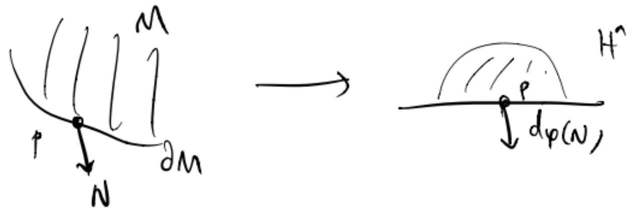
§4.4.1 Orientation on the Boundary

Definition 4.4.12: Outward Pointing Vector

Let M be an oriented manifold with boundary and let $p \in \partial M$. A vector $N \in TM_p$ is *outward pointing* if whenever

$$\phi \colon U \to V \subset H^n$$

is a chart around p, $\mathrm{d}\phi(N) \in TH^n_{\phi(N)}$ has negative last coordinate.



Definition 4.4.13: Boundary Orientation

Let M be an oriented manifold with boundary. The *boundary orientation* on ∂M is defined by declaring a basis $(v_1, \ldots, v_{n-1})$ for $T(\partial M)_p$ to be positively-oriented whenever $(N, v_1, \ldots, v_{n-1})$ is a positively-oriented basis for TM_p, for some outward pointing $N \in TM_p$.

It is left as an exercise to show this is a well-defined orientation.

Remark 4.4.14: Orientation of S^n

$S^n \subset \mathbb{R}^{n+1}$ is orientable since it is ∂B^{n+1} (closed ball), abd B^{n+1} is orientable.

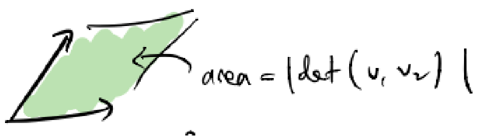
§4.5 Volume and Integration

§4.5.1 Volume of Parallelopiped

Here is a fact: If $B = (v_1, \ldots, v_n)$ is an ordered basis for $\mathbb{R}^n$, then

$$|\det(v_1, \ldots, v_n)|$$

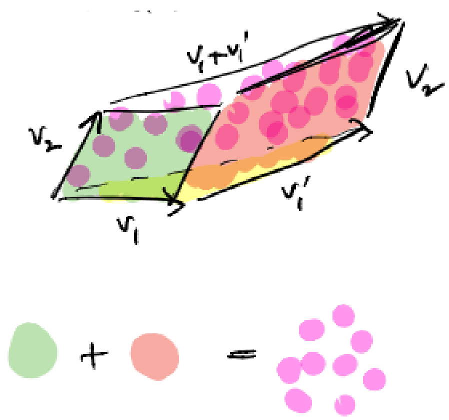
is the volume of the parallelopiped spanned by $v_1, \ldots, v_n$.



Proof of Fact. Let $B = (v_1, \ldots, v_n)$ be an ordered bases of $\mathbb{R}^n$. Consoder the function

$$B \mapsto \operatorname{sgn}(\det(v_1, \ldots, v_n)) \cdot \operatorname{vol} P_B$$

where P_B is the parallelopiped spanned by the basis vectors. This function is multilinear as can be seen:



and vanishes on linearly dependent set of vectors, so it is an alternating n-tensor. Moreover, our function takes the value 1 on the standard basis. These two facts imply that the function is the determinant function, since $\Lambda^n(\mathbb{R}^n)$ is a 1-dim vector space (hence its elements are exactly scalings of det). ■

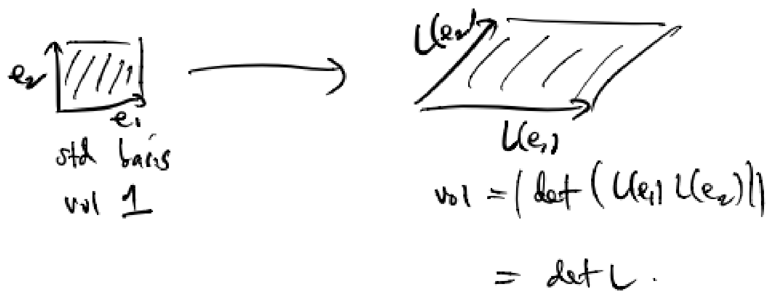
The above fact shows:

1. More generally, the only reasonable way to assign a notion of volume to a parallelopiped in a vector space is via an alternating tensor.

2. If $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear then L distorts volume by $|\det L|$, i.e.

$$\text{vol}(L(A)) = |\det L| \cdot \text{vol}(A).$$

   For instance:

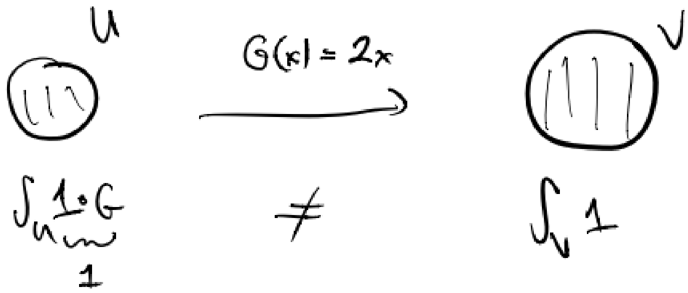


§4.5.2 The Change-of-Variables Formula

Suppose $G\colon U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, and $f\colon V \to \mathbb{R}$ is a compactly supported function. Then

$$\int_V f \mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_U f \circ G |\det \mathrm{d}G| \mathrm{d}x^1 \cdots \mathrm{d}x^n.$$

Explanation: You need some new factor above to account for the fact that G stretches volume. E.g.:



The factor $|\det \mathrm{d}G_p|$ is the infinitesimal distortion of the volume of G, since near p, $G \approx \mathrm{d}G$, and $|\det \mathrm{d}G|$ measures volume distortion of $\mathrm{d}G$.

Note that you don't integrate functions, you integrate functions against Lebesgue measure, so the Change-of-Variables formula includes a factor relating to Lebesgue measure.

§4.5.3 Integration of Forms

Definition 4.5.1: Integral of a Compactly Supported Top-Degree Form on $\mathbb{R}^n$

Suppose ω is a compactly supported n-form in some open $U \subset \mathbb{R}^n$, where

$$\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Then we define

$$\int \omega := \int f \mathrm{d}x^1 \cdots \mathrm{d}x^n.$$

Proposition 4.5.2: Integral of Pullback of Top-Deg Form via o.p./o.r. Diffeo, on $\mathbb{R}^n$

Suppose $G\colon U \to V$ is an o.p. or o.r. diffeomorphism between open sets U, V in $\mathbb{R}^n$, and ω is a compactly supported n-form on V. Then

$$\int G^*\omega = \begin{cases} \int \omega & G \text{ is o.p.} \\ -\int \omega & G \text{ is o.r.} \end{cases}$$

Proof. Suppose G is o.p., and we write $\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Then

$$\begin{aligned} \int G^*\omega &= \int f \circ G \det(\mathrm{d}G) \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n \quad \text{Pullback Formula for Top-Degree Forms} \\ &= \int f \mathrm{d}x^1 \cdots \mathrm{d}x^n \quad \text{Change-of-Variables Formula} \\ &= \int \omega. \end{aligned}$$

■

Definition 4.5.3: Integral of Cmpctly Supp Top-Deg Form on Orient Manif

Suppose M is an oriented n-manifold, and ω is an n-form on M that is compactly supported within the domain of some orientation preserving chart. Then we define

$$\int \omega := \int (\phi^{-1})^* \omega,$$

where ϕ is any such chart.

Note: the definition of $\int \omega$ does not depend on the particular ϕ, since the transition map between any two such ϕ is an orientation preserving diffeomorphism, hence by the above Proposition, the integrals are the same.

Definition 4.5.4: Integral of Any Top-Deg Form on Orient Manif

Suppose M is an oriented n-manifold, and ω is an n-form on M. Pick a partition of unity $\{\rho_i\}$ such that each ρ_i is compactly supported within the domain of some orientation preserving chart. Then we define

$$\int_M \omega := \sum_i \int \rho_i \omega.$$

To ensure this is a good definition, we need the following fact: Fact: $\int \omega$ is independent of the choice of $\{\rho_i\}$.

Proof of Fact. Suppose $\{\psi_j\}$ is another partition of unity. Then we have

$$\begin{aligned} \sum_i \int \rho_i \omega &= \sum_i \int \sum_j \psi_j \rho_i \omega \\ &= \sum_{i,j} \int \psi_j \rho_i \omega \\ &= \sum_j \int \left(\sum_i \rho_i \right) \psi)_j \omega \\ &= \sum_j \int \psi_j \omega. \end{aligned}$$

■

Remark 4.5.5

1. If ω is a 0-form on an oriented 0-manifold M, we define

$$\int \omega = \sum_{p \in M} \operatorname{sgn}(p) \omega(p)$$

where recall that $\operatorname{sgn}(p)$ is $+$ or $-$ depending on orientation.

2. If $S \subset M$ is an oriented k-submanifold. Then if ω is a k-form on M, we write

$$\int_S \omega := \int_S i^* \omega$$

where $i\colon S \to M$ is inclusion.

Proposition 4.5.6: Properties of Integrals of n-forms on n-Manifold

1. (integration is linear)

$$\int_M a\omega + g\eta = a\int_M \omega + b\int_M \eta.$$

2. If $-M$ is M with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

3. If ω is positively oriented (i.e. $\omega(v_1, \dots, v_n) > 0$ whenever $v_1, \dots, v_n$ is a positively oriented basis for some tangent space), then

$$\int \omega > 0.$$

4. If $G\colon M \to N$ is an o.p. or o.r. diffeomorphism,

$$\int_M G^*\omega = \begin{cases} \int_N \omega & G \text{ is o.p.} \\ -\int_N \omega & G \text{ is o.r.} \end{cases}$$

Theorem 4.5.7: Stokes Theorem

Let M be an oriented n-manifold with boundary, and ω be a compactly supported $(n-1)$-form. Then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Note that ∂M has orientation $(v_1, \dots, v_{n-1}) \in T\partial M_p$ is positively orientated when $(N, v_1, \dots, v_{n-1})$ is positively oriented for TM_p, and N outward pointing.

Remark 4.5.8: on Stokes' Theorem

1. If $\partial M =$, then the Theorem implies $\int_M \mathrm{d}\omega = 0$.

2. Suppose $M = [a, b]$ (so a 1-Manifold with boundary). Suppose ω is a smooth function, then $\mathrm{d}\omega\omega'(x)\mathrm{d}x$..

   PIC

   So, Stokes' Theorem says

$$\int_a^b \omega'(x)\mathrm{d}x = \omega(b) - \omega(a),$$

   which is the Fundamental Theorem of Calculus.

3. Stoke's Theorem specializes in dimensinos 2 and 3 to the classical Greene's and Stokes' Theorems from multivariable calculus.

§4.5.4 Stokes' Theorem

Stokes' Theorem. Suppose $M = H^n$, ω is supported in

$$[-R, R] \times \cdots \times [-R, R] \times [0, R],$$

write

$$\omega = \sum_j \omega_i \mathrm{d}x^1 \wedge \cdots \wedge \hat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n$$

and hence

$$\begin{aligned} \mathrm{d}\omega &= \sum_i \mathrm{d}\omega_i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \hat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^1 \wedge \cdots \wedge \hat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_i \end{aligned}$$

So

$$\int -H^n \mathrm{d}\omega = \sum_i (-1)^{i-1} \int_0^R \int_{-R}^R$$

■

§4.6 The DeRham Isomorphism Theorem