Discussion 12/10 3:00

Vectors:

angle btw vectors, direction of vectors,

length/magnitude of vectors,

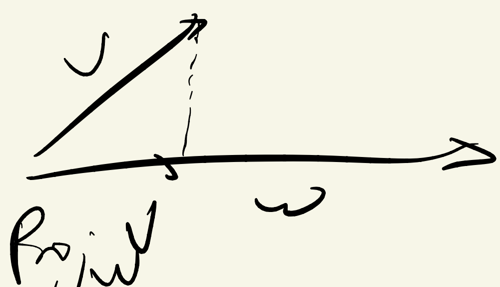
parallel vectors, perpendicular vectors

$$v \times w = 0 \iff v \parallel w$$

$$v \cdot w = 0 \iff v \perp w.$$

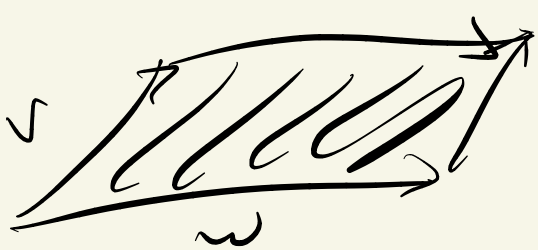
Geometric interpretation of $\cdot$ & $\times$:

$\cdot$:



$$v \cdot w = |\mathrm{Proj}_w v|\,|w|.$$

$\times$:



$v \times w$ = area.

$\{a, b, c\} \cancel{\times} \{e, f, g\}$

Equations of Lines & Planes

$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle.$

plan:

$\underbrace{\vec{n} \cdot \langle x_0, y_0, z_0 \rangle = 0}$.

Know how to tell it

lines are parallel,
planes are parallel,
a line perpendicular to a plane
crossing a given pt.

line of intersection b/w planes

$n_1 \times n_2$

Vector-Valued functions, Parametric Curves

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\boxed{\vec{r}'(t)} = \langle x'(t), y'(t), z'(t) \rangle$$

rate of change / velocity as you move along the curve.

arc-length of a curve: $a \le t \le b$

$$\int_a^b |r'(t)|\, dt.$$

arc-length parametrization $t \longmapsto s$.

$$\underbrace{s(t) = \int_0^t |r'(u)|\, du}_{\text{expression in } t.}$$

Solve for t.

$$T = \frac{r'(t)}{|r'(t)|}$$

$$k = \left|\frac{dT}{ds}\right|$$

$$= \frac{|T'(t)|}{|r'(t)|}$$

Partial derivatives

$f(x, y, z)$

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}$$
$$\| \qquad \| \qquad \|$$
$$D_{\hat{i}} f \quad D_{\hat{j}} f \quad D_{\hat{k}} f$$

Tangent planes to a surface.

$z = f(x, y)$, find T. plane at (x_0, y_0)

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Another method : using level sets.

$$\underbrace{x^2+y^2+z^2}_{F(x,y,z)} = 4$$

$$z^2 = 4 - x^2 - y^2$$

$$z = \pm\sqrt{4-x^2-y^2}$$

∇F is my normal vector.

Chain Rule:

$z = f(x, y)$, $x = h(t)$, $y = g(t)$.

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.}$$

Implicit differentiation.

Directional derivative & Gradient

$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$\boxed{D_{\vec{u}} f} = \boxed{\nabla f \cdot \hat{u}}$$

$\|$

$D_{\vec{u}} f(x_0, y_0, z_0)$ = rate of change of f at (x_0, y_0, z_0) in the dir of $\hat{u}$.

$dir(\hat{u}) = dir(\nabla f)$

∇f is the direction of fastest increase.

$\boxed{\hat{u} \perp \nabla f} = 0$ change

$dir(-\nabla f)$ direction of fastest decrease.

Min / Max

Local min/max

1st derivative test

give critical pts

2nd derivative test to give "type" of critical pt.

& Global min/max.

E.V.T.:

Global min/max exists if f is defined on a closed & bdd. set.

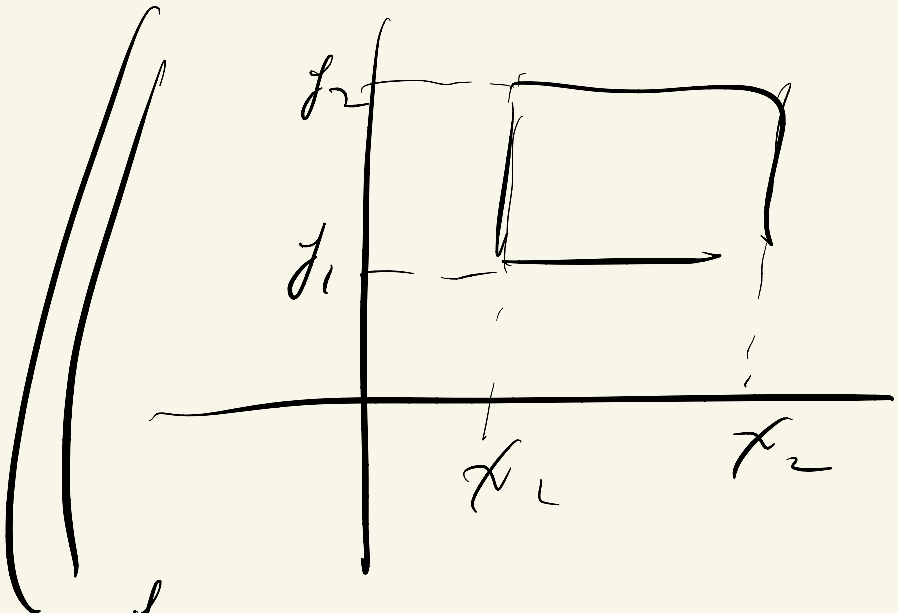
1st derivative test to find crit pts on the interior

check boundary.

Lagrange Multiplier.

Double integral

$$\iint_R f(x,y)\, dA$$



$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\, dx\, dy$$

$$=$$

$$\int_{x_1}^{x}\int_{y_1}^{y_2} f(x,y)\, dy\, dx$$

$$\iint_R f(x,y)\, dA$$ = area under $f(x,y)$ over R

Over general regions.

Type I

$c_1 \le x \le c_2$

$h_1(x) \le y \le h_2(x)$

Type II

$c_1 \le y \le c_2$

$h_1(y) \le x \le h_2(y)$

Polar Coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$$\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(\theta, r)\cdot \underline{r}\,dr\,d\theta.$$

$\left.\begin{array}{l}\text{Cylindrical Coord.}\\ \text{Spherical Coord.}\end{array}\right\}$ triple integral.

$$\iiint_R f(x,y,z)\, dA$$

$\shortparallel$

hypervolume over a 3D area.

Vector fields:

$$F(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

Conservative vector fields:

$$F = \nabla f.$$

Know how to tell if F is conservative.

If F is conservative, then

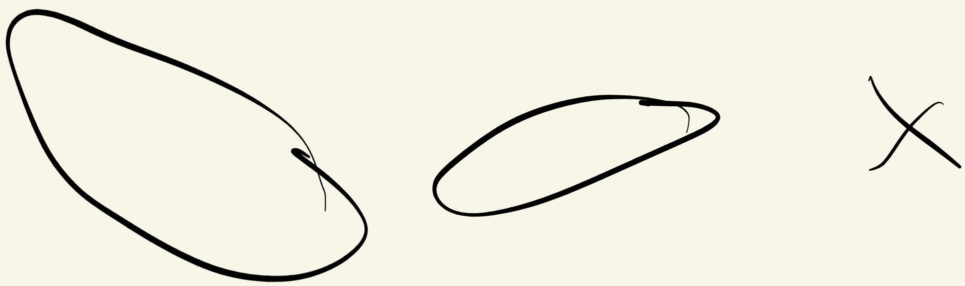
$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

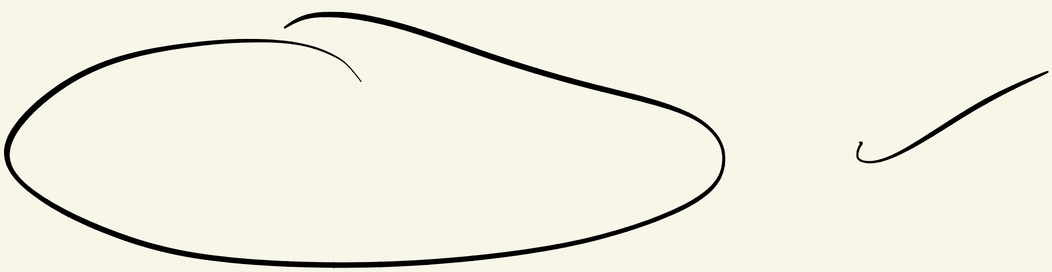
the converse: if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$,

and F is defined over an open,
and simply-connected region,
then F is conservative.

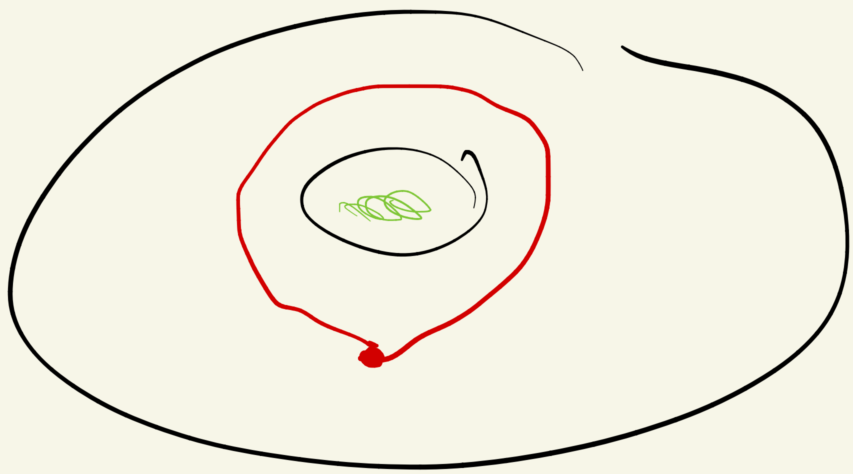
Simply-connected:

first: has to be one piece.





second: and closed loop in the region only encloses points in the region



Basically: Region has no holes.

Once you F is conservative.
you should know how to find
f.

$$F = P\,\hat{i} + Q\,\hat{j}$$

$$\underbrace{\int P(x,y)\,dx = \underline{\qquad\qquad} + \underline{g(y)}}_{\text{"}f\text{"}}$$

$$\frac{\partial f}{\partial y} = \boxed{\underline{\qquad\qquad} + g'(y)}$$

compare with Q.

Line Integrals

$$\boxed{\int_C f(x,y)\, ds}$$

C is $\vec{r}(t)$

$a \le t \le b$.

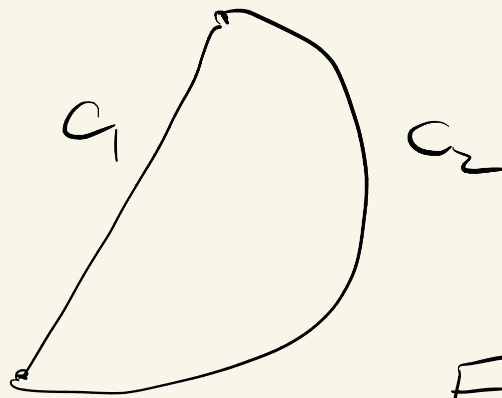
$$\int_a^b f(x(t), y(t))\, r'(t)\, dt$$

$$\int_C f(x,y)\, dx = \int f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\, dy \qquad \int \qquad '' \qquad y'(t)\, dt$$

Important:

Another thing



$F = P\,i + Q\,j$

Line integrals of vector fields.

$$\int_C F \cdot d\vec{r} = \int_C P\,dx + \int_C Q\,dy$$

$$= \int_C P\,dx + Q\,dy$$

$$\int_C \nabla f \cdot d\vec{r} = \underline{f(\vec{r}(b)) - f(\vec{r}(a))}$$

Green's Thm. C

$$\boxed{\int_C P\,dx + Q\,dy} = \boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

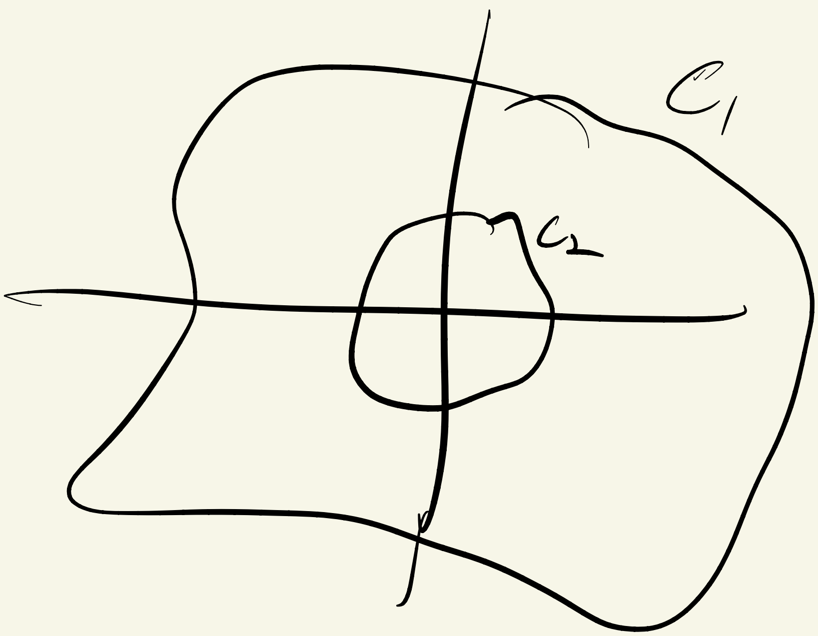
C has to be positively oriented,
simple, closed curve

Extended Version of Green's Thm.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$

$$\int_{C_1} P\,dx + Q\,dy$$

C_1 is any loop around origin



$$\int_{C_1} P\,dx + Q\,dy = -\int_{C_2} P\,dx + Q\,dy$$

$r_0 = (x_0, y_0) \quad r_1 = (x_1, y_1)$

$$\boxed{\vec{r}(t) = (1-t)\, r_0 + t\, r_1}$$

$0 \leq t \leq 1$

$t = 0 \to r_0$

$t = 1 \to r_1$

$$\int_C F \cdot d\vec{r}$$

$a \leq t \leq b.$

$C = \vec{r}(t).$

if $F = \nabla f$

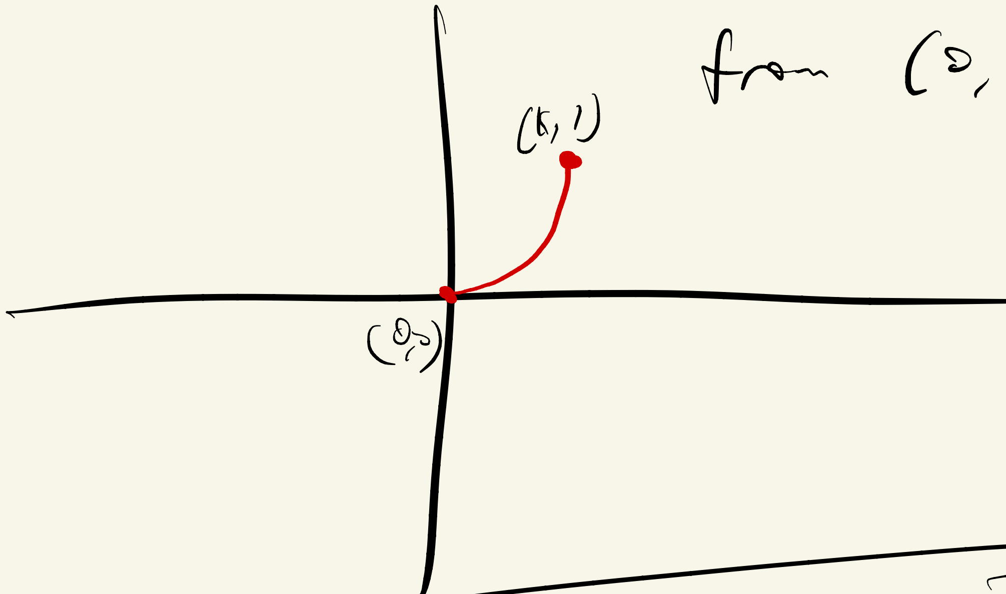
in particular

$$\int_C F \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int_C f \, ds$$

$$f(x,y) = 2x$$

C is part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$



$$\vec{r}(t) = \langle \boxed{t}, \boxed{t^2} \rangle$$

$$0 \le t \le 1$$

$$\vec{r}(x) = \langle x, x^2 \rangle_{0 \le x \le 1}.$$

$$\int_C f(x,y)\,ds \qquad f(x,y) = 2x$$

$$= \int_0^1 f(\overset{t}{x(t)}, \overset{t^2}{y(t)}) \cdot \| r'(t) \| \, dt$$

$$= \int_0^1 (2t \cdot \sqrt{1+4t^2}) \, dt$$

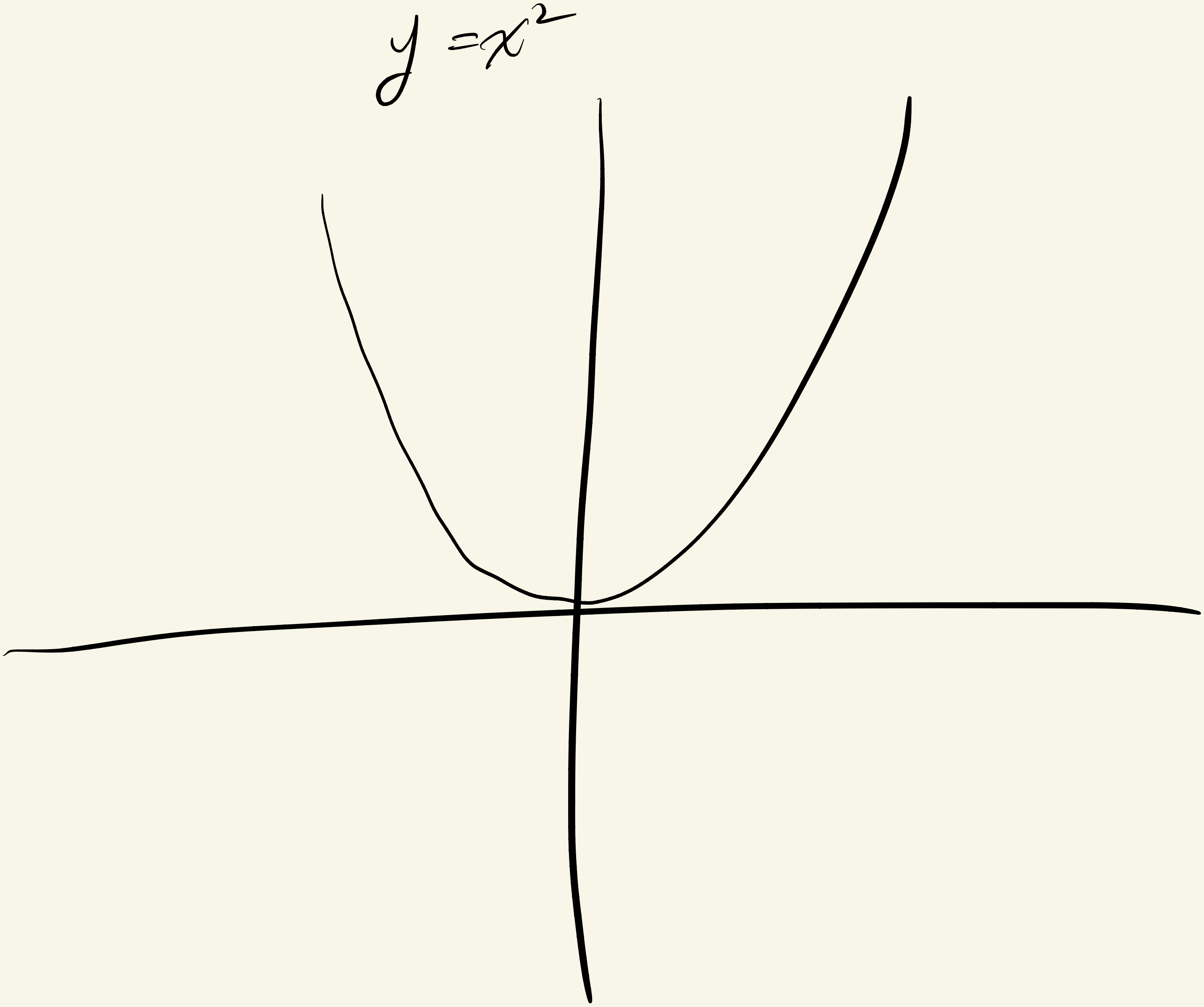
$$f(x,y) = 2x$$

$$f(t,t^2) = 2t.$$

$$r'(t) = \langle 1, 2t \rangle.$$

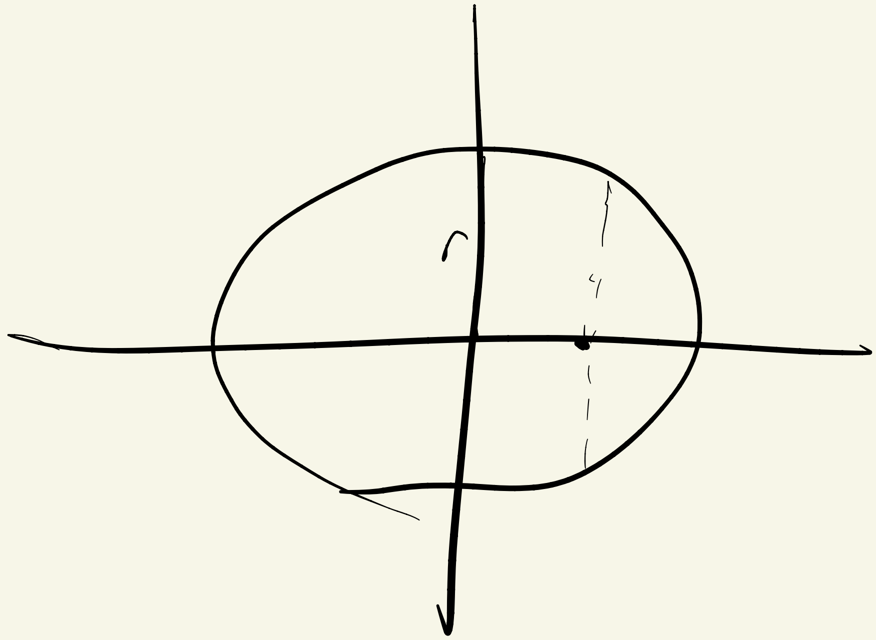
$$|r'(t)| = \sqrt{1^2 + 4t^2} = \sqrt{1+4t^2}$$

$$\Rightarrow \frac{5\sqrt{5}-1}{6}.$$



$$\vec{r}(t) = \langle t, t^2 \rangle$$

$$\vec{r}(x) = \langle x, x^2 \rangle$$



$$\vec{r}(\theta) = \langle r\cos\theta ,\ r\sin\theta \rangle$$

$$0 \le \theta \le 2\pi.$$

$y =$ [illegible]

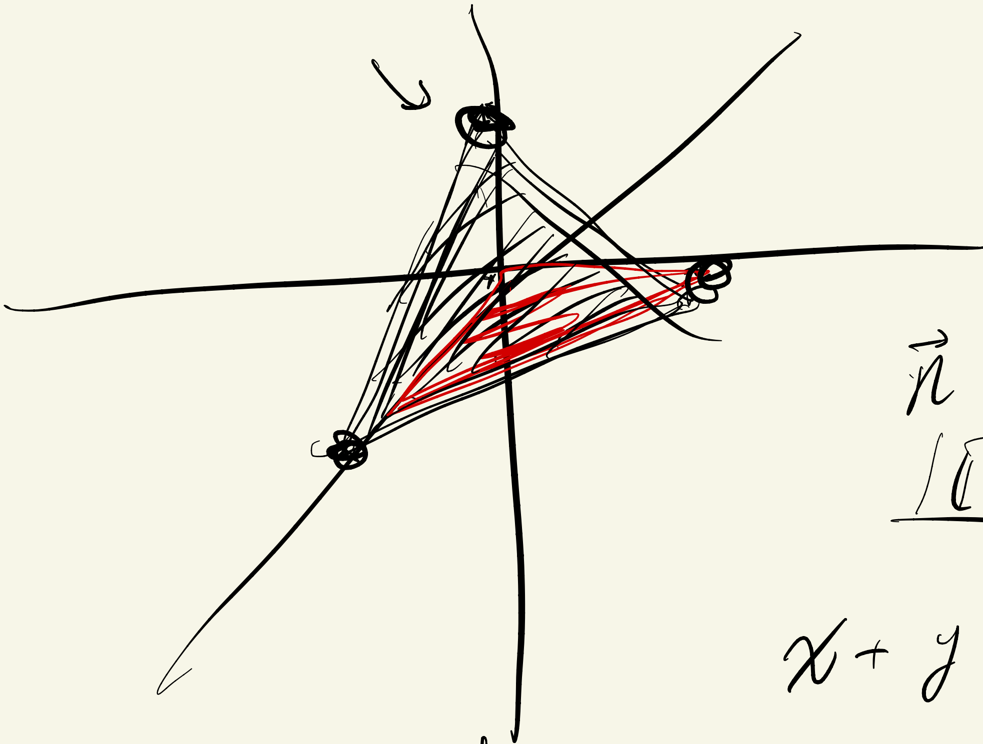
$$\langle x, \quad \rangle$$

$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

$$\langle x, \ \pm\sqrt{1 - x^2} \rangle$$



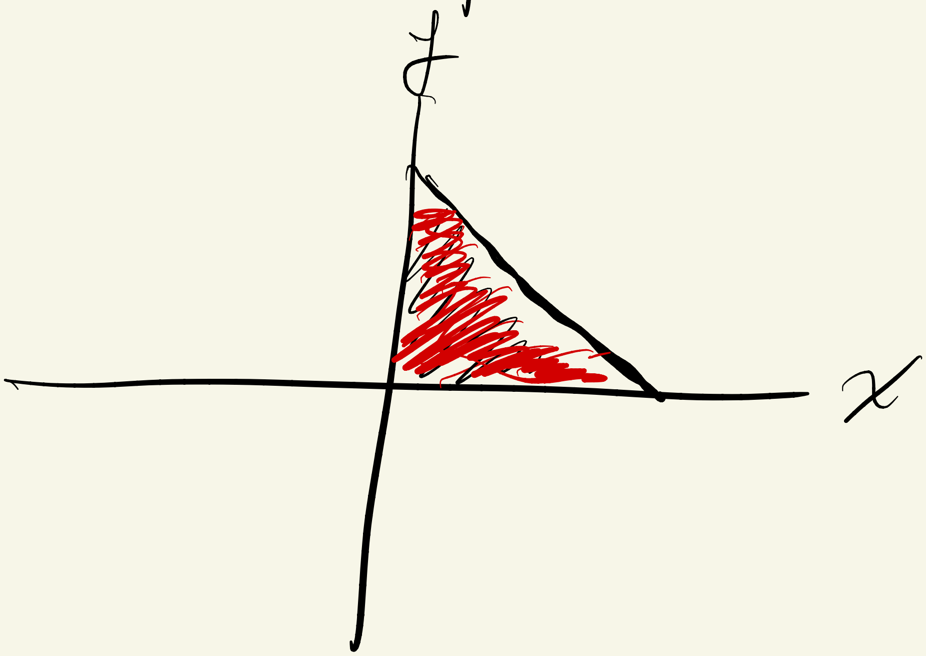
$\vec{n} = \langle 1, 1, 1 \rangle$

$\boxed{(0, 0, 1)}$

$x + y + z + d = 0$

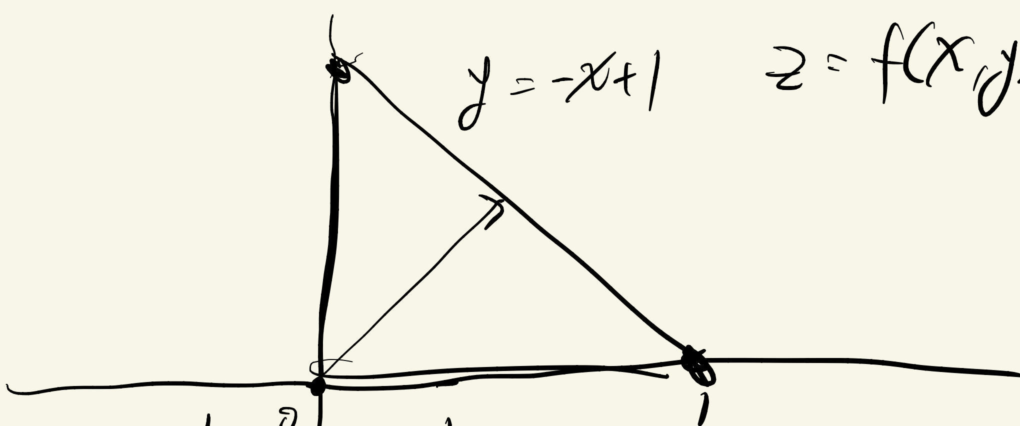
$1 + d = 0$

$d = -1$



$x + y + z - 1 = 0$

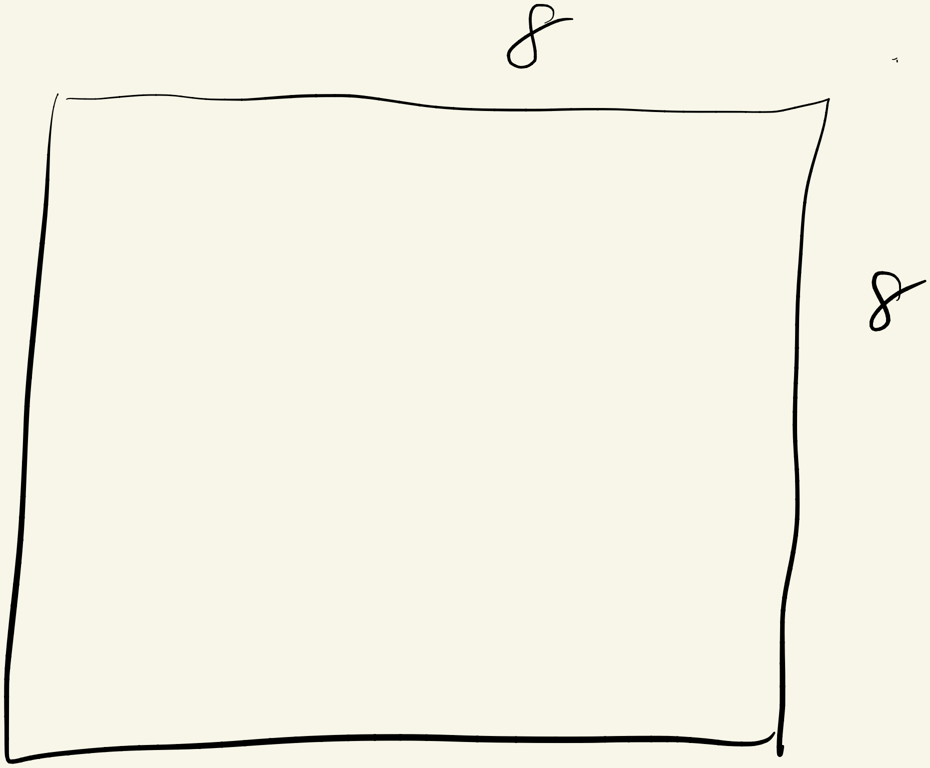
$x + y + z = 1.$



$z = f(x, y) = \underline{1 - x - y}$

$$\int_0^1 \int_0^{-x+1} (1 - x - y)\, dy\, dx$$

$$\int_D f(x,y)\, dA.$$



$y^2x\,dx + x^2y\,dy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$2xy - 2yx$

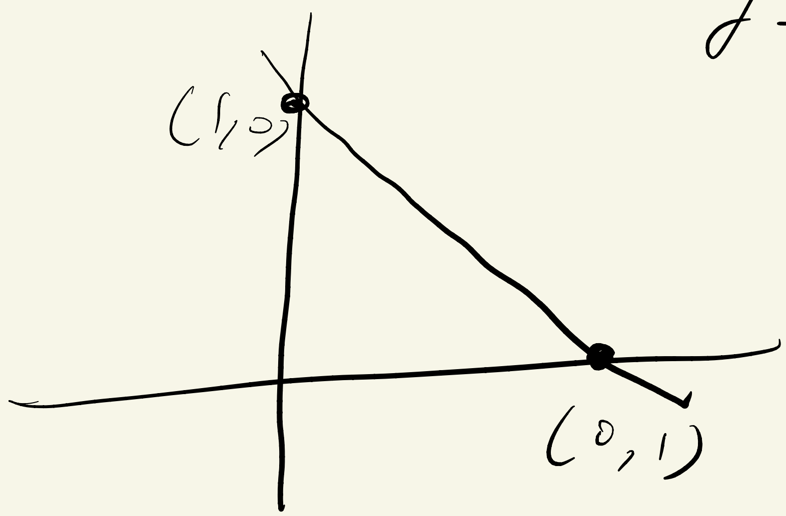
$2xy - 2yx$

$y^2x\,dx + x^2y\,dy$

$2yx -$

$$(x^2+1)^3$$

$y = -x + 1$



$$dy = \boxed{\frac{dy}{dx}}\, dx$$